

Equivariant Rozansky-Witten classes and TFTs

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Abstract

We first construct the Rozansky-Witten model coupled to BF theory and Chern-Simons theory using the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) method. Then we apply the machinery developed in some earlier papers about AKSZ theories and characteristic classes to these concrete models: the BF-Rozansky-Witten model and the Chern-Simons-Rozansky-Witten model. In the former case, we obtain characteristic classes on the target hyperKähler manifold equipped with a group action as a generalization of the original Rozansky-Witten classes. We also give the prescription for similar classes associated with a holomorphic symplectic manifold and demonstrate the invariance of such classes explicitly.

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1 Introduction

In [1] Rozansky and Witten described a three dimensional topological sigma model with the target space being a hyperKähler manifold M , and they showed that the partition function of the theory is a three-manifold invariant of finite type. The perturbative expansion of the Rozansky-Witten (RW) model gives rise to an interesting weight system which depends on the hyperKähler manifold. The weights are labeled by trivalent graphs which are just Feynman diagrams of the field theory. Roughly speaking, if we choose a complex structure J and a holomorphic symplectic form Ω on M , then we can use the Riemann curvature tensor of the hyperKähler manifold to construct the vertex

$$V_{jkl} = dz^{\bar{i}} R_{\bar{i}j}^n \Omega_{nk} \in \Omega^{0,1}(M, T^* \otimes T^* \otimes T^*) ,$$

where we have used the complex coordinates for J . Taking a trivalent graph Γ with $2n$ vertices we can contract the vertices V with Ω^{-1} according to the graph. For example, the following graph



corresponds to

$$V_{i_1 i_2 i_3} V_{j_1 j_2 j_3} V_{l_1 l_2 l_3} V_{k_1 k_2 k_3} \Omega^{i_1 j_1} \Omega^{i_2 j_2} \Omega^{j_3 l_3} \Omega^{l_1 k_1} \Omega^{l_2 k_2} \Omega^{k_3 i_3} , \quad (1.1)$$

which is a differential form in $\Omega^{0,4}(M)$. For a general trivalent graph Γ with $2n$ vertices the resulting form will be in $\Omega^{0,2n}(M)$, and this form is $\bar{\partial}$ -closed. Indeed it should be understood as an element of the Dolbeault cohomology group $H_{\bar{\partial}}^{0,2n}(M)$, and we refer to this class as a RW characteristic class. The crucial properties of a RW class associated to Γ is that it depends on Γ only through its homology class and it is invariant under deformations of the hyperKähler metric. Since its discovery in 1996 many different aspects of the RW classes and invariants have been studied, and in fact the RW classes can be defined for any holomorphic symplectic manifold [2, 3]. We refer the reader to [4] for a nice mathematical summary of RW theory. Recently in [5, 6] a treatment of the RW model within the Batalin-Vilkovisky (BV) formalism has been developed. The BV framework offers very elegant and natural explanations of many properties of RW theory. Actually, there is a canonical relation between a wide class of 3-dimensional TFTs and characteristic classes of graded (super) manifolds. The RW theory is just one particular manifestation of this generic phenomena.

In the present work our goal is to construct the equivariant version of RW-classes for hyperKähler manifolds with a compatible group action. We address this problem as physicists by constructing appropriate extensions of the original RW model and by studying their perturbative expansion. The extension of the RW model we consider is a “gauged” version of the RW model. Namely, we show that if the holomorphic symplectic manifold admits a Hamiltonian group action then we can couple the RW model to BF-theory or, upon some additional condition on moment map, to Chern-Simons (CS) theory. Our analysis is very similar to the ideas presented in [6] and the present work can be regarded as a concrete illustration for the general ideas advertised in [6]. Previously the RW model coupled to CS theory was written down and studied by Kapustin and Saulina in [7] and it is related to the topological twist of the Gaiotto-Witten model [8]. By assuming that there is a holomorphic moment map on the hyperKähler manifold, one can enlarge the BRST transformations of the RW model with extra terms involving the group action. Here we will construct the same model with a more streamlined and uniform approach known as the AKSZ construction. Due to the need to make the BRST transformations nilpotent, an extra requirement was imposed on the moment map. This requirement restricts the applicability of the model. On the other hand, the RW model coupled to BF theory is more liberal, and also as we shall see, the perturbative expansion of such models only has a finite number of terms, in sharp contrast to the CS-RW case.

Let us briefly summarize the perturbative results coming from BF-RW theory. The RW model can be formulated on a manifold M with a holomorphic symplectic form Ω , and we assume that there is a group acting holomorphically and preserving Ω . Assuming that the group action is Hamiltonian with holomorphic moment map μ_A we can write down the RW model coupled to BF theory whose 3-valent vertex V_{ijk} is

$$V_{jkl} = dz^{\bar{i}} R_{\bar{i}j}^n \Omega_{nk} + A^A \nabla_j \nabla_k \partial_l \mu_A + A^A k_A^i R_{ij}^n \Omega_{nk} \in \Omega^{0,1}(M, T^* \otimes T^* \otimes T^*) \oplus \wedge^1 \mathfrak{g}^* \quad (1.2)$$

where we assume symmetrization in j, k, l and $k_A^i = \partial_j \mu_A \Omega^{ji}$. Here \mathfrak{g}^* is the dual of the Lie algebra of the Lie group acting on M and A^A is basis element in \mathfrak{g} of odd degree. In order to write down V we have to pick up a connection on M compatible with J and Ω (which always exists). For a hyperKähler manifold we can just pick the Levi-Civita connection for the hyperKähler metric and the last term in the vertex V_{jkl} is identically zero. By taking a trivalent graph Γ with $2n$ vertices we contract the V 's using Ω^{-1} in the same fashion as in the RW story. The resulting object

$$c_\Gamma \in \bigoplus_{p+q=2n} \Omega^{0,p}(M) \otimes \wedge^q \mathfrak{g}^* \quad (1.3)$$

is closed under the following differential

$$Q = \bar{\partial} + A^A k_A^i \partial_i - \frac{1}{2} f_{AB}^C A^A A^B \frac{\partial}{\partial A^C} .$$

We can define the corresponding cohomology group $H_Q^{2n}(M)$ and understand c_Γ as a cohomology class. The class satisfies the following properties:

- it is invariant under a deformation of the connection (hyperKähler metric) on M
- it depends on the graph Γ only through its graph homology class.

These classes give rise to a weight system for the perturbative expansion of the BF-RW model. At each order \hbar^n of perturbation, there is a collection of 3-valent graphs as dictated by Wick's theorem. The partition function (without integrating over zero modes) has the form

$$\sum_{\Gamma} b_\Gamma c_\Gamma , \quad (1.4)$$

where the b_Γ 's depend on the three manifold and are the same as in the RW model. This expansion has a finite number of terms and the number of terms depends on the dimensionality of M . Many nice properties of the expansion (1.4) follow from manipulations using the BV machinery, which we will review later. In sections 2,3,4, we restrict ourselves to M being hyperKähler to simplify the formulae. The generalization to any holomorphic symplectic manifold with a Hamiltonian group action is given in section 5.

A similar analysis can be performed for the CS-RW model. However, there are additional requirements on the moment maps and the perturbative expansion continues to all orders. There still exists an interpretation in terms of appropriate characteristic classes.

The paper is organized as follows: In section 2 we first construct the RW, BF-RW and CS-RW models using a systematic approach within the BV framework. In particular, the CS-RW model is identical with the one constructed by Kapustin and Saulina in [7]. In section 3, we show that the new construction allows us to do computations with superfields

and that leads to a tremendous simplification in the perturbative computation and we obtain the same results as those obtained with the more traditional component approach. We present the Feynman rules for the RW, BF-RW and CS-RW models. In section 4 we give an interpretation of the perturbative partition function for these models in terms of appropriate characteristic classes. We show heuristically why these classes are closed and independent of certain data. In section 5 some formal and mathematical issues are collected and discussed. In particular we discuss the general case of a holomorphic symplectic manifold with a Hamiltonian group action. Section 6 gives a summary of the results and some outlook.

2 Systematic Construction of the RW, BF-RW and CS-RW Model

2.1 Review of the AKSZ construction of the RW Model

Topological field theories have since long been a very effective tool in producing and studying topological invariants. Typically for a TFT whose fields Φ are mappings

$$\Phi: \Sigma \rightarrow \mathcal{M},$$

the action of the model can usually be written down using totally canonical data on Σ and \mathcal{M} . In particular, no metric data is used. As a result, the expectation values of gauge invariant operators are expected to depend only on these canonical data and nothing else. Here we name Σ and \mathcal{M} as the source and target space respectively, both may be super (graded) manifolds. In [9] a conceptually clear approach for constructing topological field theories was introduced. Besides its easy-to-use feature, this approach meshes well with the BV formalism, and therefore offers one the ability to discuss the gauge dependence of the theory in a way that the BRST formalism cannot. This approach is known under the name of the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction. Using geometrical data on Σ and \mathcal{M} , the AKSZ approach gives a canonical solution of the classical master equation within the BV-framework. For a general discussion and examples of the AKSZ approach, we refer to [10, 11, 5, 12, 13] and more references therein. We will not review the general construction here nor will we use the language of graded manifolds in any essential way, but rather focus on some specific 3D TFTs.

Our source manifold is $\Sigma = T[1]\Sigma$, where Σ is a 3-manifold with coordinate x^a . We can assemble forms on Σ into superfields by introducing a formal odd (degree 1) variable θ^a which transforms as dx^a . Therefore a polyform on Σ is just a function of θ^a, x^a ¹:

$$f(x, \theta) = f(x) + \theta^a f_a(x) + \frac{1}{2} \theta^a \theta^b f_{ab}(x) + \frac{1}{6} \theta^a \theta^b \theta^c f_{abc}(x).$$

Here $f(x)$ is a 0-form, $f_a(x)$ is a 1-form etc. Often we denote $f(x)$ as $f_{(0)}$, $dx^a f_a(x)$ as $f_{(1)}$ and so on (also for the zero form component if no confusion is likely, we even drop the

¹The convention here is slightly different from [6, 5] in that now the product of superfields is just the wedge product.

subscript $_{(0)}$). The multiplication of superfields directly corresponds to the wedge product and the de Rham differential is $D = \theta^a \partial_a$.

Our target manifold \mathcal{M} is a degree 2 graded symplectic manifold, which means that there is a symplectic structure ω of degree 2. Written in local Darboux coordinates

$$\omega = \frac{1}{2} \omega_{AB} d\Phi^A \wedge d\Phi^B ,$$

where $\omega_{AB} = -(-1)^{|\Phi^A||\Phi^B|} \omega_{BA}$ and Φ^A is the coordinate on \mathcal{M} with $|\Phi^A|$ denoting its degree. The BV space will be the space of superfields $\Phi^A = \Phi^A(x, \theta)$, *i.e.* space of maps², $\text{Maps}(T[1]\Sigma, \mathcal{M})$. There is an odd symplectic form on the BV space:

$$\omega_{BV} = \frac{1}{2} \int d^6 z \, \omega_{AB} \, \delta \Phi^A \delta \Phi^B , \quad (2.1)$$

where $d^6 z = d^3 x d^3 \theta$ and if any function is written in bold then we assume that it is a superfield. With such an odd symplectic form, one can define a naive Laplacian

$$\Delta = \int d^3 x \, (\omega^{-1})^{AB} (-1)^{|\Phi^A|} \left(\frac{\partial}{\partial \Phi_{(3)}^A} \frac{\partial}{\partial \Phi_{(0)}^B} + \frac{\partial}{\partial \Phi_{(1)}^A} \frac{\partial}{\partial \Phi_{(2)}^B} \right) . \quad (2.2)$$

The Laplacian induces an odd Poisson bracket in the BV space in the following way

$$\{f, g\} = (-1)^{|f|} \Delta(fg) - (-1)^{|f|} (\Delta f)g - f(\Delta g) , \quad f, g \in \text{Maps}(T[1]\Sigma, \mathcal{M}) . \quad (2.3)$$

There is an important identity

$$\left\{ \int d^6 z \, f(\Phi), \int d^6 z \, g(\Phi) \right\} = - \int d^6 z \, \{f, g\}(\Phi) , \quad (2.4)$$

which relates the bracket in the BV space to the bracket in the target space \mathcal{M} .

The path integral is defined not over all BV space, but rather over a Lagrangian submanifold in the BV space on which (2.1) vanishes. The choice of this Lagrangian submanifold is called the *gauge fixing*. The following key statement is about gauge invariance in the BV framework: if a function \mathcal{O} is annihilated by the Laplacian (2.2), then the integral of \mathcal{O} is invariant under small changes of the Lagrangian submanifold \mathcal{L} . Furthermore the integral of something Δ -exact is zero. This says that the action of any theory must satisfy $\Delta e^S = 0$, or equivalently $\Delta S + 1/2 \{S, S\} = 0$. This equation is known as the quantum master equation. In fact, usually the classical master equation $\{S, S\} = 0$ and the quantum one are fulfilled simultaneously.

The main innovation of [9] is to encode the classical master equation (which is on the space of mappings) into a single function Θ on the target space \mathcal{M} with the property $\{\Theta, \Theta\} = 0$ (this is non-trivial only when Θ is of odd degree). The property of Θ would ensure

$$\left\{ \int d^6 z \, \Theta, \int d^6 z \, \Theta \right\} = - \int d^6 z \, \{\Theta, \Theta\} = 0 .$$

²This space requires proper mathematical definition, e.g. see [11] for a discussion. However this is not relevant to our considerations and we work constantly in local coordinates.

The term Θ will be used as the interaction term for the theory, we can also write a kinetic term. Suppose there is a Liouville form $\Xi_A d\Phi^A$ such that $d\Xi = \omega$, then the kinetic term is

$$S_{kin} = \int d^6 z \, \Xi_A(\Phi) D\Phi^A.$$

Notwithstanding the fact that Ξ may only exist locally, the kinetic term is well defined if $\partial\Sigma = \emptyset$. We will constantly use the Darboux coordinates where $\Xi_A = \Phi^B \Omega_{BA}$.

The full action is then given by

$$S = S_{kin} + S_{int} = \int d^6 z \, \left(\frac{1}{2} \Phi^A \Omega_{AB} D\Phi^B + \Theta \right).$$

The action automatically satisfies $\{S, S\} = 0$ and $\Delta S = 0$.

The hard and non-canonical part is naturally the gauge fixing. The rough idea is that, sometimes there are quite 'natural' choices of \mathcal{L} , but the resulting action has too much residue gauge symmetry rendering the path integral ill defined. So we perturb \mathcal{L} a bit in order to fix those residue gauge symmetry.

Let us take a look at the Rozansky-Witten model and see how to arrive at it from our approach. Let M be a complex manifold admitting a holomorphic symplectic form Ω . Denote the target space of the RW model as the graded manifold $\mathcal{M}_{RW} \equiv T^{*0,1}[2](T^{*0,1}[1]M)$, which is locally parameterized by the following: $X^i, X^{\bar{i}}$ coordinates of M ; $p_{\bar{i}}$ degree 2 fiber coordinates in the anti-holomorphic cotangent direction; $v^{\bar{i}}$ degree 1 fiber coordinate of $T^{0,1}M$; $q^{\bar{i}}$ degree 1 fiber coordinate of $T^{*0,1}M$. We choose the following symplectic form of degree 2 on \mathcal{M}_{RW}

$$\omega_{RW} = dp_{\bar{i}} dX^{\bar{i}} + dq_{\bar{i}} dv^{\bar{i}} + \frac{1}{2} \Omega_{ij} dX^i dX^j, \quad (2.5)$$

where Ω_{ij} is the holomorphic 2-form on M and we allocate degree 2 for Ω (alternatively we can introduce formal parameter of degree 2 in front of Ω). The natural choice of Hamiltonian function of degree 3 is

$$\Theta = -p_{\bar{i}} v^{\bar{i}},$$

which satisfies $\{\Theta, \Theta\} = 0$, and it acts as $\bar{\partial}$ on functions of $f(X, v)$. With these data, the BV action is given by

$$S_{RW} = \int d^6 z \, \left(p_{\bar{i}} D X^{\bar{i}} + q_{\bar{i}} D v^{\bar{i}} + \frac{1}{2} \Omega_{ij} X^i D X^j - p_{\bar{i}} v^{\bar{i}} \right). \quad (2.6)$$

By construction this action satisfies the classical master equation on the BV space $\text{Maps}(T[1]\Sigma, \mathcal{M}_{RW})$.

The only data we used so far is the holomorphic symplectic structure on M . Now we will discuss the gauge fixing. For the sake of clarity and simplicity of formulae we specialize now to the hyperKähler case, then Ω_{ij} is covariantly constant with respect to the Levi-Civita connection Γ_{ik}^j and $\Gamma_{\bar{i}\bar{k}}^{\bar{j}}$. However, we would like to stress that this restriction

is not essential and everything can be carried out for a generic holomorphic symplectic manifold (see section 5). Skipping some algebra, the Lagrangian submanifold is given by the set of conditions

$$\begin{aligned}\tilde{p}_{\bar{i}(0)} &= \tilde{p}_{\bar{i}(1)} = \tilde{p}_{\bar{i}(2)} = X_{(2)}^i = X_{(3)}^i = \mathbf{q}_{\bar{i}} = 0 , \\ \tilde{p}_{\bar{i}(3)} &= \frac{1}{6} \Omega_{lj} R_{i\bar{i}}^{\bar{j}} X_{(1)}^l \wedge X_{(1)}^i \wedge X_{(1)}^k .\end{aligned}\quad (2.7)$$

Note that we have defined a new variable

$$\tilde{\mathbf{p}}_{\bar{i}} = \mathbf{p}_{\bar{i}} + \Gamma_{i\bar{k}}^{\bar{j}} \mathbf{q}_{\bar{j}} \mathbf{v}^{\bar{k}} ,$$

which transforms as a tensor in contrast to $\mathbf{p}_{\bar{i}}$. Also, the components of fields are now defined using covariant derivatives, e.g. $v_{ab}^{\bar{i}} = 1/2[\nabla_{\theta^b}, \nabla_{\theta^a}]\mathbf{v}^{\bar{i}}|_{\theta=0}$. The reader may consult [5] for more details.

One may check that this set of conditions set ω_{BV} to zero. Evaluated on this Lagrangian submanifold, one finds the action

$$S_{RW} = \frac{1}{2} \int d^3x \left(\Omega_{ij} X_{(1)}^i \wedge d^\nabla X_{(1)}^j - \frac{1}{3} R_{k\bar{k}}^{\bar{i}} X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j v^{\bar{k}} \right) , \quad (2.8)$$

where $d^\nabla X_{(1)}^i = dX_{(1)}^i + \Gamma_{jk}^i dX_{(0)}^j X_{(1)}^k$. We notice that there is no kinetic term for the fields $X_{(0)}^i, X_{(0)}^{\bar{i}}, v^{\bar{i}}$, and the kinetic term for $X_{(1)}$ is not invertible, since the de Rham operator has an infinite dimensional kernel. We use the freedom of deforming the Lagrangian submanifold to obtain a nice quadratic term. The safest way is to deform every field by $\delta\phi = \{\Psi, \phi\}$, where ϕ is any field in the theory and Ψ is a cleverly chosen function. We notice that this kind of deformation will maintain the condition $\omega_{BV}|_{\mathcal{L}+\delta\mathcal{L}} = 0$, since the deformation is generated by a Hamiltonian vector field. For the RW model we choose

$$\Psi = -\frac{1}{2} \left(g_{i\bar{j}} X_{(1)}^i \wedge *dX_{(0)}^{\bar{j}} \right) .$$

Applying Ψ to deform the Lagrangian submanifold defined by (2.7), we get the complete action

$$\begin{aligned}S_{RW_{gauged}} &= \frac{1}{2} \int d^3x \left(g_{i\bar{j}} dX_{(0)}^i \wedge *dX_{(0)}^{\bar{j}} - g_{i\bar{j}} X_{(1)}^i \wedge *d^\nabla v_{(0)}^{\bar{j}} \right. \\ &\quad \left. + \Omega_{ij} X_{(1)}^i \wedge d^\nabla X_{(1)}^j - \frac{1}{3} R_{k\bar{k}}^{\bar{i}} X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j v^{\bar{k}} \right) ,\end{aligned}\quad (2.9)$$

which agrees with the theory constructed originally by Rozansky and Witten in [1].

According to the general prescription of BV-AKSZ [9], the BRST transformations for the theories are always obtained by calculating $\delta_{BRST}\phi = \{S, \phi\}|_{\mathcal{L}}$. For the RW model we find

$$\begin{aligned}\delta X^{\bar{i}} &= v^{\bar{i}} , \quad \delta X^i = 0 , \\ \delta X_{(1)}^i &= dX^i , \\ \delta v^{\bar{i}} &= 0 .\end{aligned}$$

Note that $\{S, S\} = 0$ off shell, but the BRST transformation is the restriction of $\{S, \cdot\}$ onto \mathcal{L} and may only close on-shell in general. In the RW model the transformations close off-shell.

2.2 BF-RW Model

In this section, we will couple the RW model to gauge fields. Let us assume that the holomorphic symplectic manifold (M, J, Ω) underlying the RW model admits an action of a Lie group \mathbf{G} . Let us assume that this action preserves the complex structure J and the holomorphic symplectic structure Ω . At infinitesimal level the action is realized by the vector fields k_A such that the Lie brackets are given by

$$[k_A, k_B] = f_{AB}^C k_C, \quad (2.10)$$

and we assume that $\mathcal{L}_{k_A} J = 0$ and $\mathcal{L}_{k_A} \Omega = 0$. Thus in complex coordinates we can write $k_A^\mu \partial_\mu = k_A^i(z) \partial_i + k_A^{\bar{i}}(\bar{z}) \partial_{\bar{i}}$. Finally let us assume that the action is Hamiltonian with respect to Ω , *i.e.* there exist a holomorphic moment map μ_A defined by

$$\partial_j \mu_A = k_A^i \Omega_{ij}, \quad (2.11)$$

which is equivariant

$$\{\mu_A, \mu_B\}_\Omega = f_{AB}^C \mu_C, \quad (2.12)$$

where $\{, \}_\Omega$ is Poisson bracket with respect to the holomorphic symplectic form Ω . Next we introduce the graded manifold $\mathcal{M}_{BF-RW} \equiv (\mathbf{g}^*[1] \otimes \mathbf{g}[1]) \times \mathcal{M}_{RW}$ with the coordinates of $\mathbf{g}^*[1] \otimes \mathbf{g}[1]$ being B_A and A^A of degree 1 and \mathcal{M}_{RW} was defined in the previous subsection. This space is equipped with an even symplectic form of degree 2

$$\omega_{BF-RW} = \omega_{RW} + \delta B_A \delta A^A,$$

where ω_{RW} is defined in (2.5). The Hamiltonian of degree 3 is defined as follows

$$\Theta = -p_{\bar{i}} v^{\bar{i}} + \frac{1}{2} f_{AB}^C A^A A^B B_C + \mu_A A^A,$$

where μ_A is of degree 2 (since Ω was assumed to be degree 2) The corresponding master equation is satisfied

$$\{\Theta, \Theta\} = f_{AB}^D f_{DE}^F A^A A^B A^E B_F + (\{\mu_A, \mu_B\}_\Omega - f_{AB}^C \mu_C) A^A A^B = 0,$$

since the first term vanishes due to the Jacobi identity and second due to the equivariance of the moment maps. Also we use the fact that the moment map μ_A is holomorphic, *i.e.* $\bar{\partial} \mu_A = 0$.

With this data it is straightforward to apply the AKSZ construction. On the space $\text{Maps}(T[1]\Sigma, \mathcal{M}_{BF-RW})$ there is a BV-bracket defined by the following odd symplectic structure

$$\omega_{BV} = \int d^6 z \left(\delta B_A \delta A^A + \delta p_{\bar{i}} \delta X^{\bar{i}} + \delta q_{\bar{i}} \delta v^{\bar{i}} + \frac{1}{2} \Omega_{ij} \delta X^i \delta X^j \right). \quad (2.13)$$

The master action defining the BF-RW model is thus

$$S_{BF-RW} = \int d^6 z \left(\mathbf{B}_A D \mathbf{A}^A + \mathbf{p}_{\bar{i}} D \mathbf{X}^{\bar{i}} + \mathbf{q}_{\bar{i}} D \mathbf{v}^{\bar{i}} + \frac{1}{2} \Omega_{ij} \mathbf{X}^i D \mathbf{X}^j + \frac{1}{2} f_{AB}^C \mathbf{A}^A \mathbf{A}^B \mathbf{B}_C - \mathbf{p}_{\bar{i}} \mathbf{v}^{\bar{i}} + \mu_A \mathbf{A}^A \right). \quad (2.14)$$

This model is defined for any holomorphic symplectic manifold with a Hamiltonian action of a group.

Now to proceed to the gauge fixing, we again specialize to the hyperKähler case and we choose the Lagrangian submanifold for the RW sector in the same way as in (2.7). For the gauge sector, we use the metric on Σ_3 to Hodge decompose the differential forms on Σ_3 . That is to say, we decompose any superfield \mathbf{A} (and similarly for \mathbf{B}) according to

$$\mathbf{A} = \mathbf{A}^h + \mathbf{A}^e + \mathbf{A}^c = \mathbf{A}^h + \theta^a \partial_a \mathbf{r} - \nabla^a \partial_{\theta^a} \mathbf{s},$$

where c, e, h stands for coexact, exact and harmonic; \mathbf{r}, \mathbf{s} are some superfields and $-\nabla^a \partial_{\theta^a}$ is just d^\dagger written in the super language. If one decomposes the above equation into components, one gets exactly the usual Hodge decomposition.

The symplectic BV-form is thus decomposed as

$$\int d^6 z \delta \mathbf{B}_A \delta \mathbf{A}^A = \int d^6 z \delta(\mathbf{B}_A)^c \delta(\mathbf{A}^A)^e + \delta(\mathbf{B}_A)^e \delta(\mathbf{A}^A)^c + \delta(\mathbf{B}_A)^h \delta(\mathbf{A}^A)^h.$$

We do the gauge fixing by choosing the following Lagrangian submanifold: $(\mathbf{B}_A)^e = (\mathbf{A}^A)^e = 0$ and for the harmonic sector we set $A_{(3)}^h = A_{(2)}^h = B_{(3)}^h = B_{(2)}^h = 0$. We find the complete action for the gauge-fixed BF-RW model to be

$$\begin{aligned} S_{BFRW} &= S_{RW \text{ gauged}} + S_{BF} + S_{new} \\ S_{RW \text{ gauged}} &= \frac{1}{2} \int \left(\Omega_{ij} X_{(1)}^i \wedge D^A X_{(1)}^j - \frac{1}{3} R_{\bar{k}\bar{i}}^i X_{(1)}^{\bar{k}} \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j v^{\bar{k}} \right) \\ S_{BF} &= \int \left((B_{(1)A} \wedge dA_{(1)}^A + \frac{1}{2} f_{AB}^C A_{(1)}^A \wedge A_{(1)}^B \wedge B_{(1)C}) - B_{(2)C} \wedge (dA_{(0)}^C + f_{AB}^C A_{(1)}^A A_{(0)}^B) \right. \\ &\quad \left. + A_{(2)}^A \wedge (-dB_{(0)A} - f_{AB}^C A_{(1)}^B B_{(0)C} + f_{AB}^C A_{(0)}^B B_{(1)C}) \right) \\ S_{new} &= \int \left(\partial_i \mu_A X_{(1)}^i \wedge A_{(2)}^A - \frac{1}{6} (\nabla_i \nabla_j \partial_k \mu_A) X_{(1)}^i \wedge X_{(1)}^j \wedge X_{(1)}^k A_{(0)}^A \right), \end{aligned} \quad (2.15)$$

where $D^A X_{(1)}^i = d^\nabla X_{(1)}^i + (\nabla_j k_A^i) A_{(1)}^A \wedge X_{(1)}^j$. For the fields $A_{(0,1,2)}, B_{(0,1,2)}$ in this expression, the gauge condition has been imposed even though we do not make explicit the superscript h or c . This also explains the absence of $A_{(3)}, B_{(3)}$. One can rewrite this action in more familiar form related to the Faddeev-Popov trick by introducing Lagrange multipliers which enforce the Lorentz gauge for one-forms and rewriting 2-forms explicitly as d^\dagger of another field.

One can also deform the gauge choice by using $\Psi = -1/2 g_{i\bar{j}} X_{(1)}^i \wedge *D^A X_{(1)}^{\bar{j}}$ as we did in the previous subsection. The derivation of S_{new} in (2.15) is very similar to the CS-RW model, and we will present some technical details in the next subsection.

2.3 CS-RW model

In this subsection we construct the CS-RW model within the AKSZ approach and we will show that upon specific gauge fixing it is the same model discussed previously in [7] within the BRST framework.

For the CS-RW model, let us define the target manifold as the following graded manifold

$$\mathcal{M}_{CS-RW} \equiv \mathfrak{g}[1] \times \mathcal{M}_{RW} . \quad (2.16)$$

If the algebra \mathfrak{g} is equipped with an *ad*-invariant metric η_{AB} then on \mathcal{M}_{CS-RW} there is a symplectic form of degree 2

$$\omega_{CS-RW} = \omega_{RW} + \frac{1}{2} \eta_{AB} \delta A^A \delta A^B , \quad (2.17)$$

where A^B are coordinates of degree 1 on \mathfrak{g} . Let us assume as in the previous subsection that M admits the holomorphic Hamiltonian action with moment maps μ_A . Let us consider the following Hamiltonian function of degree 3 on \mathcal{M}_{CS-RW}

$$\Theta = -p_{\bar{i}} v^{\bar{i}} + \frac{1}{6} f_{ABC} A^A A^B A^C + \mu_A A^A , \quad (2.18)$$

where $f_{ABC} = f_{AB}{}^D \eta_{DC}$ and μ_A is of degree 2 (since Ω was assumed to be degree 2). In order to fulfill the master equation, we need

$$\{\Theta, \Theta\} = -\frac{1}{4} f_{AB}{}^C f_{CDE} A^A A^B A^D A^E + (\{\mu_A, \mu_B\}_\Omega - f_{AB}{}^C \mu_C) A^A A^B - \mu_A \mu_B \eta^{AB} = 0 . \quad (2.19)$$

Compared to the discussion in previous subsection, this gives us an extra constraint on the moment maps, namely $\mu_A \mu_B \eta^{AB} = 0$. Despite the fact that this additional condition looks exotic there are examples, see the discussion in [7].

Now on the space $\text{Maps}(T[1]\Sigma, \mathcal{M}_{CS-RW})$ there is the following BV symplectic form

$$\omega_{BV} = \int d^6 z \left(\frac{1}{2} \eta_{AB} \delta \mathbf{A}^A \delta \mathbf{A}^B + \delta \mathbf{p}_{\bar{i}} \delta \mathbf{X}^{\bar{i}} + \delta \mathbf{q}_{\bar{i}} \delta \mathbf{v}^{\bar{i}} + \frac{1}{2} \Omega_{ij} \delta \mathbf{X}^i \delta \mathbf{X}^j \right) . \quad (2.20)$$

The action defining the CS-RW model is thus given by

$$\begin{aligned} S_{CSRW} = \int d^6 z \left(\frac{1}{2} \eta_{AB} \mathbf{A}^A D \mathbf{A}^A + \mathbf{p}_{\bar{i}} D \mathbf{X}^{\bar{i}} + \mathbf{q}_{\bar{i}} D \mathbf{v}^{\bar{i}} + \frac{1}{2} \Omega_{ij} \mathbf{X}^i D \mathbf{X}^j \right. \\ \left. + \frac{1}{6} f_{ABC} \mathbf{A}^A \mathbf{A}^B \mathbf{A}^C - \mathbf{p}_{\bar{i}} \mathbf{v}^{\bar{i}} + \mu_A \mathbf{A}^A \right) . \end{aligned} \quad (2.21)$$

This action automatically satisfies the master equation provided that there is a holomorphic Hamiltonian group action on the holomorphic symplectic manifold with the additional property³ on the moment maps $\mu_A \mu_B \eta^{AB} = 0$.

³It is interesting to point out that the same condition appears in Gaiotto-Witten work [8] and it is related to the Chern-Simons theory for super-Lie algebra.

Next let us discuss the gauge fixing of the present model. Again, we specialize to the hyperKähler case for simplicity. We will now demonstrate that the above AKSZ model is indeed equivalent to the model obtained in [7]. The symplectic form of the RW sector and CS sector decouple, so we may choose the gauge fixing for the RW sector just as before, while for the CS sector, we set the exact part of each component to zero $A_{(1)}^e = A_{(2)}^e = A_{(3)}^e = 0$ and also $A_{(3)}^h = A_{(2)}^h = 0$. We obtain the standard CS action plus the RW sector (2.9) and a new part from expanding

$$S_{new} = \int d^6 z \left(\mu_A(\mathbf{X}^i) \mathbf{A}^A \right) .$$

Performing the grassmanian integral, we find

$$\int d^3 x \left(-\frac{1}{6} (\nabla_i \nabla_j \partial_k \mu_A) X_{(1)}^i X_{(1)}^j X_{(1)}^k A_{(0)}^A - \frac{1}{2} \Omega_{ij} \nabla_k k_A^i X_{(1)}^k X_{(1)}^j A_{(1)}^A + \partial_i \mu_A X_{(1)}^i A_{(2)}^A \right) . \quad (2.22)$$

In general, if we have an action on the manifold which preserves the metric⁴, the vector field generating the action satisfies the Killing equation $\nabla_{(\mu} k_{\nu)}^A = 0$, where $k_\nu^A = \eta^{AB} g_{\nu\mu} k_B^\mu$. From this equation and the Bianchi identity for the Riemann tensor it follows that

$$\nabla_\mu \nabla_\nu k_\rho^A = -R_{\rho\mu}{}^\lambda{}_\nu k_\lambda^A . \quad (2.23)$$

On a hyper-Kähler manifold, with an action preserving the holomorphic symplectic form and the Kähler form, we can obtain a stronger relation for $k_A^{\bar{i}}$. In [7] it was shown that under these circumstances we have

$$(\nabla_k \nabla_j \partial_i \mu_A) X_a^k X_b^j X_c^i A_{(0)}^A = -\Omega_{kl} R_{i\bar{k}}{}^l{}_j X_a^k X_b^j X_c^i k_A^{\bar{k}} A_{(0)}^A . \quad (2.24)$$

Using this relation together with (2.22) and (2.9), we find the total action to be

$$\begin{aligned} S &= S_{CS} + S_{RW} + S_{new} , \\ S_{CS} &= \int \left(\frac{1}{2} \eta_{AB} A_{(1)}^A \wedge dA_{(1)}^B - \eta_{AB} A_{(2)}^A \wedge dA_{(0)}^B \right. \\ &\quad \left. + \frac{1}{6} f_{ABC} A_{(1)}^A \wedge A_{(1)}^B \wedge A_{(1)}^C - f_{ABC} A_{(2)}^A \wedge A_{(1)}^B \wedge A_{(0)}^C \right) , \\ S_{RW} &= \frac{1}{2} \int \left(\Omega_{ij} X_{(1)}^i \wedge d^\nabla X_{(1)}^j - \frac{1}{3} R_{k\bar{k}}{}^i{}_j X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j v^{\bar{k}} \right) , \\ S_{new} &= \int \left(\frac{1}{6} R_{k\bar{k}}{}^i{}_j X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j k_A^{\bar{k}} A_{(0)}^A \right. \\ &\quad \left. - \frac{1}{2} \Omega_{ki} \nabla_j k_A^k X_{(1)}^i \wedge X_{(1)}^j \wedge A_{(1)}^A + \partial_i \mu_A X_{(1)}^i \wedge A_{(2)}^A \right) , \end{aligned} \quad (2.25)$$

where $d^\nabla X_{(1)}^i = dX_{(1)}^i + \Gamma_{jk}^i X_{(1)}^j dX_{(0)}^k$. We see that the combination $v_{(0)}^{\bar{i}} - k_A^{\bar{i}} A_{(0)}^A$ naturally appears and it is useful to introduce the field $\eta^{\bar{i}} = v_{(0)}^{\bar{i}} - k_A^{\bar{i}} A_{(0)}^A$. Furthermore, we can

⁴It is important to stress that we need the condition $\mathcal{L}_{k_A} g = 0$ only to match with the results from [7]. In further discussion of the construction of characteristic classes we do not have to assume any special properties of metric or connection under the group action.

define a "big" covariant derivative by $D^A X_{(1)}^i = d^\nabla X_{(1)}^i + \nabla_j k_A^i A_{(1)}^A \wedge X_{(1)}^j$. This derivative is covariant under both gauge transformations and changes of coordinates. With these definitions, the final form of the action is

$$S_{CS} + S_{RW} + S_{new} = S_{CS} + \frac{1}{2} \int \left((\Omega_{ij} X_{(1)}^i \wedge D^A X_{(1)}^j - \frac{1}{3} R_{k\bar{k}}{}^i{}_j X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j \eta^{\bar{k}} + \partial_i \mu_A X_{(1)}^i \wedge A_{(2)}^A) \right). \quad (2.26)$$

We can analyze the BRST-transformations related to this gauge fixing in the same way as for the RW model. At the level of components, the BRST transformations are given by

$$\begin{aligned} \delta X_{(0)}^{\bar{i}} &= v_{(0)}^{\bar{i}}, \quad \delta X_{(0)}^i = k_A^i A_{(0)}^A, \\ \delta X_{(1)}^i &= dX_{(0)}^i - k_A^i A_{(1)}^A - (\partial_j k_A^i) X_{(1)}^j A_{(0)}^A, \\ \delta A_{(0)}^A &= -\frac{1}{2} f_{BC}{}^A A_{(0)}^B A_{(0)}^C - \eta^{AB} \mu_B, \\ \delta A_{(1)}^A &= dA_{(0)}^A + f_{BC}{}^A A_{(1)}^B A_{(0)}^C + \eta^{AB} \partial_i \mu_B X_{(1)}^i, \\ \delta v_{(0)}^{\bar{i}} &= 0. \end{aligned} \quad (2.27)$$

In order to come in contact with [7], let us rewrite using $\eta^{\bar{i}} = v_{(0)}^{\bar{i}} - k_A^{\bar{i}} A_{(0)}^A$. We find

$$\delta \eta^{\bar{i}} = -\partial_{\bar{j}} k_A^{\bar{i}} v_{(0)}^{\bar{j}} A_{(0)}^A + \frac{1}{2} f_{BC}{}^A A_{(0)}^B A_{(0)}^C k_A^{\bar{i}} + \eta^{AB} k_A^{\bar{i}} \mu_B = -\partial_{\bar{j}} k_A^{\bar{i}} \eta^{\bar{j}} A_{(0)}^A + \eta^{AB} k_A^{\bar{i}} \mu_B$$

with the help of (2.10). With the field $\eta^{\bar{i}}$, the relevant changes to the BRST transformation are

$$\begin{aligned} \delta X_{(0)}^{\bar{i}} &= \eta^{\bar{i}} + k_A^{\bar{i}} A_{(0)}^A, \\ \delta \eta_{(0)}^{\bar{i}} &= -\partial_{\bar{j}} k_A^{\bar{i}} \eta^{\bar{j}} A_{(0)}^A + \eta^{AB} k_A^{\bar{i}} \mu_B, \end{aligned}$$

while the rest stays the same. Comparing with section 2.3 in [7], we find the exact same BRST transformations, and the same action. In order to get kinetic terms for $X_{(0)}$ we add the standard BRST-exact term

$$\delta \left(g_{i\bar{j}} X_{(1)}^i \wedge *d^A X_{(0)}^{\bar{j}} \right). \quad (2.28)$$

This will also produce a kinetic term for the co-exact part of $X_{(1)}^i$ and for the scalar $\eta^{\bar{i}}$, just like in the RW-model.

The authors of [7] also added the term

$$\delta \left(\sqrt{h} g_{i\bar{j}} k_A^i \bar{\mu}_B \eta^{AB} \eta^{\bar{j}} \right), \quad (2.29)$$

where h is the determinant of the metric on the source, and $\xi^i = k_A^i \bar{\mu}_B \eta^{AB}$.

We have now reproduced the action and BRST transformations obtained in [7] within the AKSZ framework. In [7] they have a general function $f(A)$ and a Lagrange multiplier

field B whose equation of motion enforces $f(A) = 0$. Our gauge fixing corresponds to $f(A) = \nabla^a A_a^A$.

Next our goal is to study the perturbative partition function for RW, BF-RW and CS-RW models. The gauge fixing of the BV models we have discussed so far may be useful for the interpretation of the models, however it is not very practical for doing perturbative calculations. In the next section we will discuss a gauge fixing for the actions (2.14) and (2.21) at the level of superfields and this will allow us to have simple Feynman rules. By general arguments presented in the coming sections, the results of the calculation will correspond to certain characteristic classes of some differential Q on the target space. But before specializing to concrete models, we will go through some general features of the perturbation theory for 3D AKSZ models.

3 Perturbation Expansion of the Models

The perturbative expansion of the models we have discussed gives invariants both for the source manifold Σ and for the target manifold \mathcal{M} . To gain insight into these invariants, it pays off to reorganize the calculations slightly. We use the RW model as an example, after that we will realise that a more straightforward gauge fixing can make the computation much simpler.

The action for the RW model (2.9) is

$$S_{RW} = \frac{1}{2} \int d^3x \left(g_{i\bar{j}} dX_{(0)}^i \wedge *dX_{(0)}^{\bar{j}} - g_{i\bar{j}} X_{(1)}^i \wedge *d^\nabla v_{(0)}^{\bar{j}} \right. \\ \left. + \Omega_{ij} X_{(1)}^i \wedge d^\nabla X_{(1)}^j - \frac{1}{3} R_{k\bar{k}}^i X_{(1)}^k \wedge \Omega_{li} X_{(1)}^l \wedge X_{(1)}^j v_{(1)}^{\bar{k}} \right).$$

Since the boson quadratic term is BRST-exact (recall that it comes from $-\delta\Psi$), the path integral localises on the zero of the quadratic term $dX^i = dX^{\bar{i}} = 0$, namely constant maps. One can therefore expand the coordinate as $X^{i(\bar{i})}(x) = x_0^{i(\bar{i})} + \xi^{i(\bar{i})}(x)$ and treat $x_0^{i,\bar{i}}$ as parameters. Note, by nature of such an expansion the zero mode of ξ should be set to zero by hand. Using some counting arguments one realises that $v^{\bar{i}}$ never participates in the perturbation theory actively other than that it sets the exact component of $X_{(1)}^i$ to zero. Furthermore there are only 3-valent vertices (one can see [1] for the argument, but we will arrive at this conclusion later in a more transparent way).

The curvature term is a ready made 3-valent vertex, and there is another one arising from expanding the connection in the second term

$$g_{i\bar{i}} \Gamma_{j\bar{k}}^{\bar{i}} dX^{\bar{j}} v^{\bar{k}} \wedge *X_{(1)}^i \rightarrow \nabla_l (g_{i\bar{i}} \Gamma_{j\bar{k}}^{\bar{i}}) \xi^l d\xi^{\bar{j}} v^{\bar{k}} \wedge *X_{(1)}^i = g_{i\bar{i}} R_{l\bar{j}}^{\bar{i}} \xi^l d\xi^{\bar{j}} v^{\bar{k}} \wedge *X_{(1)}^i. \quad (3.1)$$

There are also other 3-valent vertices, but they contain an excess of ξ^i and will not contribute.

From the kinetic terms we have the propagators, which we denote as

$$(\Omega^{-1})^{ij} H_{ab}(x_1, x_2) = \langle X_a^i(x_1) X_b^j(x_2) \rangle, \quad g^{i\bar{j}} G(x_1, x_2) = \langle \xi^i(x_1) \xi^{\bar{j}}(x_2) \rangle.$$

Things still look rather complicated, but a sleight of hand can make them easier. Define

$$X_{ab}^i = -(\Omega^{-1})^{ij} g_{j\bar{j}} \sqrt{\hbar} \partial_c \xi^{\bar{j}} \epsilon^c_{ab} , \quad \text{or} \quad X_{(2)}^i = -(\Omega^{-1})^{ij} g_{j\bar{j}} * d\xi^{\bar{j}} .$$

Using this notation, the vertex (3.1) becomes

$$R_{l\bar{k}}^j \xi^l \Omega_{jm} X_{(2)}^m v^{\bar{k}} X_{(1)}^i ,$$

which is begging to be combined with the other 3-valent vertex into a superfield form. Also notice

$$\langle X_{(2)}^i(x_1) \xi^j(x_2) \rangle = -(\Omega^{-1})^{ik} g_{k\bar{k}} * d_1 \langle \xi^{\bar{k}}(x_1) \xi^j(x_2) \rangle = -(\Omega^{-1})^{ij} * d_1 G(x_1, x_2) ,$$

which would be exactly the propagator between $X_{(2)}^i$ and ξ^j had there been a kinetic term $\Omega_{ij} X_{(2)}^i d\xi^j$.

Taking advantage of this observation, we can define a superfield

$$\xi^i = \xi^i + \theta^a (X_a^i)^c + \frac{1}{2} \theta^a \theta^b X_{ab}^i ,$$

and assemble the kinetic terms into the super form

$$S_{kin} = \int d^6 z \frac{1}{2} \xi^i \Omega_{ij} D \xi^j ,$$

the two vertices into a single term

$$V = \int d^6 z \frac{1}{6} R_{ij}^m \Omega_{mk} \xi^j \xi^k \xi^l v^{\bar{i}} ,$$

and finally the component propagators into a super propagator

$$(\Omega^{-1})^{ij} G(x_1, \theta_1; x_2, \theta_2) = \langle \xi^i(x_1, \theta_1), \xi^j(x_2, \theta_2) \rangle . \quad (3.2)$$

In this way, the perturbative expansion can be done with the simplified action

$$\int D\xi \exp \int d^6 z \left(\frac{1}{2} \xi^i \Omega_{ij} D \xi^j + \frac{1}{6} R_{\bar{i}i}^j \xi^i \Omega_{jl} \xi^l v^{\bar{i}} \right) . \quad (3.3)$$

In this formula only ξ participates in the perturbation theory, and from the way ξ^i is made, it has only a co-exact part and therefore the kinetic term is invertible. The fields $v^{\bar{i}}$ are treated as parameters.

This simplification is perhaps already foreseen by the reader, since the RW model can be constructed through the AKSZ construction which uses superfields. And from the BV action (2.6) we can get the above simplified action using another gauge choice, as follows. By looking at the action (2.6), one realizes that $\mathbf{p}_{\bar{i}}, \mathbf{X}^{\bar{i}}$ are merely spectator fields, since $p_{\bar{i}}$ appears linearly in the action. The same happens to $\mathbf{q}_{\bar{i}}$ and $\mathbf{v}^{\bar{i}}$. The only active field is $\mathbf{X}^{\bar{i}}$, and we would like to choose a gauge fixing condition for this field so that a superfield computation is possible.

The idea is to use Hodge decomposition and decompose every field into harmonic, exact and co-exact components. Obviously we can perform the Hodge decomposition at the level of superfields. \mathcal{L} is then chosen to set to zero all exact components of the superfields. However, such a decomposition for the coordinate \mathbf{X}^i is improper, since the coordinates do not have a linear structure. To fix it, we choose a basis point x_0 as before and a tangent vector ξ^i at x_0 . But the expansion cannot be the simple minded $\mathbf{X}^i = x_0^i + \xi^i$, but rather the geodesic exponential map (here we use a connection with purely holomorphic indices)

$$X^i \stackrel{\text{exp}}{=} x_0^i + \xi^i - \frac{1}{2} \Gamma_{jk}^i \xi^j \xi^k + \dots = \exp \left\{ \xi^i \partial_{x_0^i} - \xi^i \xi^j \Gamma_{ij}^k \partial_{\xi^k} \right\} x_0^i .$$

The symplectic form (2.5) is pulled back by the exponential map to

$$\exp^* \omega_{RW} = \delta p_{\bar{i}} \delta X^{\bar{i}} + \delta q_{\bar{i}} \delta v^{\bar{i}} + \frac{1}{2} \delta \xi^i \Omega_{ij} \delta \xi^j - \delta X^{\bar{i}} \delta \Theta_{\bar{i}} , \quad (3.4)$$

where

$$\Theta_{\bar{i}}(\xi) = \sum_{n=3}^{\infty} \Theta_{\bar{i}n} , \quad \Theta_{\bar{i}n} = \frac{1}{n!} \nabla_{l_4} \dots \nabla_{l_n} R_{\bar{i}l_1}{}^k{}_{l_3} \Omega_{kl_2} \xi^{l_1} \dots \xi^{l_n} . \quad (3.5)$$

If the curvature of the manifold is of type (1,1) then the covariant derivatives above commute with each other. The action (2.6) is pulled back as

$$\exp^* S = \int d^6 z \left(p_{\bar{i}} + \Theta_{\bar{i}} \right) D X^{\bar{i}} + \frac{1}{2} \xi^i \Omega_{ij} D \xi^j + q_{\bar{i}} D v^{\bar{i}} - p_{\bar{i}} v^{\bar{i}} .$$

Since now the momentum dual to $X^{\bar{i}}$ is $p_{\bar{i}} + \Theta_{\bar{i}}$, we should make the change of variable $p_{\bar{i}} \rightarrow p_{\bar{i}} + \Theta_{\bar{i}}$:

$$\exp^* S = \int d^6 z \left(p_{\bar{i}} D X^{\bar{i}} + \frac{1}{2} \xi^i \Omega_{ij} D \xi^j + q_{\bar{i}} D v^{\bar{i}} - p_{\bar{i}} v^{\bar{i}} + \Theta_{\bar{i}} v^{\bar{i}} \right) .$$

Note that $-p_{\bar{i}} v^{\bar{i}} + \Theta_{\bar{i}} v^{\bar{i}}$ generates the vector field $(v^{\bar{i}} \bar{\partial}_{\bar{i}} + v^{\bar{i}} \{ \Theta_{\bar{i}}, \cdot \})$.

We see that the first term in Θ is just the interaction term in (3.3) and we will argue later that only this first term matters. Θ satisfies the Maurer-Cartan equation

$$\bar{\partial}_{[\bar{i}} \Theta_{\bar{j}]} = -\{ \Theta_{\bar{i}}, \Theta_{\bar{j}} \} . \quad (3.6)$$

Let us now forget about the spectator fields and focus on the action⁵

$$\int d^6 z \left(\frac{1}{2} \xi^i \Omega_{ij} D \xi^j + \Theta_{\bar{i}} v^{\bar{i}} \right) .$$

Now this theory lives in a linear space and Hodge decomposition can be carried out. We set the exact part of ξ to zero. This clearly defines a Lagrangian submanifold, and the action thus restricted is

$$\int d^6 z \left(\frac{1}{2} (\xi^i)^c \Omega_{ij} D (\xi^j)^c + \Theta_{\bar{i}} (\xi^c) v^{\bar{i}} \right) , \quad (3.7)$$

⁵Note this action fails the master equation by a $\bar{\partial}$ -exact term, due to (3.6).

where $v^{\bar{i}}$ is merely a parameter.

Several other observations can further simplify the formalism. These observations apply to any 3 dimensional AKSZ model. First by using integral by part, we get

$$\int d^6 z \, \boldsymbol{\xi}^c(z) \boldsymbol{\xi}^c(z) = 0 ,$$

where we have thrown away a surface term. So the smallest non-trivial vertex will be 3-valent. Moreover, for a term with n vertices, the number of propagators p will be given by

$$p = \frac{3}{2}n . \quad (3.8)$$

This is simply due to the fact that the propagator is quadratic in the θ 's, and we must saturate all the θ integration. Since p is integer, the number of vertices n must be even.

Let now k_i be the valency of the i -th vertex. Since we must connect all the legs sticking out of a vertex with a propagator, we have

$$\frac{\sum_{i=1}^n k_i}{2} = p \Rightarrow \sum_{i=1}^n k_i = 3n , \quad (3.9)$$

where we have used (3.8). Since $k_i \geq 3 \, \forall i$, we draw the conclusion that $k_i = 3 \, \forall i$.

Finally, there will be no tadpoles, since if we connect a vertex with itself we get identically zero since we are contracting two legs of a vertex, which are symmetric (resp. anti-symmetric) with the symplectic form, which is anti-symmetric (resp. symmetric).

In summary, the only vertices which will contribute will be 3-valent, they will be even in number, and there will be no tadpoles. We have reproduced the counting argument of Rozansky and Witten but in a quicker way. For the readers who feel a certain qualm about the viability of the action (3.7), we hope we have demonstrated that the perturbation theory of (3.7) is exactly the same as the action (2.9), which is a perfectly healthy physical theory.

3.1 Perturbative Expansion of the BF-RW and CS-RW Model

In fact a similar analysis can be applied to general 3D AKSZ models. First we have to apply the exponential map in order to reduce problem to a linear space and then we can use the Hodge decomposition for the gauge fixing at the level of superfields. The counting argument about trivalent graphs and the absence of tadpoles presented in previous subsection are applicable to general 3D AKSZ models. Thus this gauge fixing scheme restricts all the Feynman diagrams to 3-valent graphs with no tadpoles. The statement may change however, when the source manifold has $H^1(\Sigma_3) \neq 0$ and when external legs are allowed (the external legs are occupied by the harmonic modes). But in this paper we consider only $\Sigma_3 = S^3$.

Since the analysis is straightforward we skip the detailed derivation of perturbative expansion for BF-RW and CS-RW models and give just a summary. The perturbation

expansion for the BF-RW model is rather simple. One sees from the kinetic term of the gauge sector that \mathbf{A} can be connected to \mathbf{B} and the only Feynman diagram involving \mathbf{A}, \mathbf{B} nontrivially is fig.1. In this figure, an incoming arrow represents \mathbf{A} and an outgoing

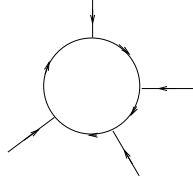


Figure 1: The only diagram involving BA propagator

one is \mathbf{B} . The external legs are occupied by $A_{(0)}^h$. Note that a harmonic zero form is but a constant. Hence technically this diagram is still two valent and thus vanishes. As a consequence, the \mathbf{A} can only appear as $A_{(0)}^h$ tagged onto the 3-valent vertex from the term $\mu_A(A^A)^h$, in the same way that $(v^{\bar{i}})^h$ is attached to the curvature term. Finally we end up with the following vertices

$$\left(R_{\bar{i}j}{}^n \Omega_{nk}(x_0) v_{(0)}^{\bar{i}} + \nabla_j \nabla_k \partial_l \mu_A(x_0) A_{(0)}^A \right) \xi^j \xi^k \xi^l, \quad (3.10)$$

where $v_{(0)}^{\bar{i}}$ and $A_{(0)}^A$ are harmonic zero modes and we choose not to integrate over them. The ξ 's will be connected by the propagator (3.2) according to the trivalent graphs without tadpoles. The resulting answer for partition function (without integration over zero modes) will be

$$\sum_{\Gamma} b_{\Gamma} c_{\Gamma}(x_0, v_{(0)}^{\bar{i}}, A_{(0)}^A), \quad (3.11)$$

where b_{Γ} is a kinematic factor (including the combinatorics) coming from the integration of the propagator (3.2) according to the trivalent graph Γ and they are obviously exactly the same as in RW model. The factor $c_{\Gamma}(x_0, v_{(0)}^{\bar{i}}, A_{(0)}^A)$ is made by contracting the vertex (1.2) with Ω^{ij} according to the graph Γ . The zero modes $A_{(0)}$ and $v_{(0)}^{\bar{i}}$ are both odd and thus the above perturbation expansion will terminate at some order.

Now let us discuss briefly the perturbative expansion for CS-RW model. In contrast with BF-RW model, the gauge field \mathbf{A} participates non-trivially in the perturbation theory of the CS-RW model and the perturbative expansion continues to all orders. We will end up with the following vertices in CS-RW theory:

$$\frac{1}{6} \left(R_{\bar{i}j}{}^n \Omega_{nk}(x_0) v_{(0)}^{\bar{i}} + \nabla_j \nabla_k \partial_l \mu_A(x_0) A_{(0)}^A \right) \xi^j \xi^k \xi^l, \quad (3.12)$$

$$\frac{1}{2} \nabla_i \partial_j \mu_A(x_0) \mathbf{A}^A \xi^i \xi^j, \quad (3.13)$$

$$\frac{1}{6} f_{ABC} \mathbf{A}^A \mathbf{A}^B \mathbf{A}^C, \quad (3.14)$$

where the superfields \mathbf{A} and ξ are assumed to be co-exact and $v_{(0)}^{\bar{i}}, A_{(0)}^A, x_0$ are constant zero modes. Now in addition to the propagator (3.2) we have to introduce a super-propagator

for \mathbf{A}

$$(\eta^{-1})^{AB} G(x_1, \theta_1; x_2, \theta_2) = \langle \mathbf{A}^A(x_1, \theta_1), \mathbf{A}^B(x_2, \theta_2) \rangle , \quad (3.15)$$

where G is the same as in (3.2). Again we end up with the following perturbative expansion for the partition function (without the integration over zero modes)

$$\sum_{\Gamma} b_{\Gamma} c_{\Gamma}(\mathbf{x}_0, v_{(0)}^{\bar{i}}, A_{(0)}^A) , \quad (3.16)$$

where c_{Γ} are constructed by contracting three vertices presented above by η^{AB} and Ω^{ij} according to the trivalent graph Γ . Now we have to look at all possible Γ with different vertices (3.12)-(3.14) and the sum (3.16) contains all orders now. However, the vertex (3.12) will enter the diagrams in very restrictive fashion since the zero modes $v_{(0)}^{\bar{i}}$ and $A_{(0)}^A$ are odd. Later we will give concrete example of c_{Γ} .

4 AKSZ Models and Characteristic Classes

We have come to the second part of the paper, where we shall discuss the mathematical aspects of our models, in particular the interpretation of the perturbative partition function. To be reasonably self contained, we shall review quickly the heuristic ideas of our approach developed in previous work. For more detailed discussion the reader may consult [6].

To construct characteristic classes for flat bundles, it is very convenient to use the Chern-Weil homomorphism applied to flat connections [14, 15]. Let \mathbf{g}^* be the dual of a Lie algebra for some Lie group \mathbf{G} . One can form the Chevalley-Eilenberg (CE) complex $(\wedge \mathbf{g}^*, \delta)$ as follows: supposing $c^q \in \wedge^{q+1} \mathbf{g}^*$ and A_0, A_1, \dots are elements of the Lie algebra basis, then δc^q , when evaluated on A_0, A_1, \dots , can be written as

$$\delta c^q(A_0, \dots, A_q, A_{q+1}) = \sum_{i < j} (-1)^{i+j+1} c^q([A_i, A_j], A_0, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_{q+1}) .$$

If one is given a flat connection A for some principle bundle (meaning A is some Lie algebra valued one form satisfying $dA + \frac{1}{2}[A, A] = 0$), one can evaluate the cochain c^q on the connection and obtain a $(q+1)$ -form. The de Rham differential acting on such a form gives

$$dc^q(\underbrace{A, \dots, A}_{q+1}) = -\frac{q+1}{2} c^q([A, A], \underbrace{A, \dots, A}_{q-1}) = -\frac{1}{q+2} \delta c^q(\underbrace{A, \dots, A}_{q+2}) .$$

In other words, if c^q is δ -closed, the resulting form by plugging in A is closed, and further it can be shown that if one changes the flat connection, the form varies by an exact form. This is the Chern-Weil homomorphism.

In the super language, the de Rham operator is written as $v^{\mu} \partial_{\mu}$, and can be regarded as a vector field on the graded manifold $T[1]M$ (where v^{μ} serves as the degree 1 coordinate of the fibre of TM). In general, d can be replaced by any nilpotent vector field Q on a graded

manifold, and the flatness condition can be replaced by an appropriate Maurer-Cartan equation: $QA + 1/2[A, A] = 0$. We can then repeat everything in the construction above and obtain classes that are Q -closed, the Q -characteristic classes.

The AKSZ models contain both ingredients of the above construction: a cocycle and a flat connection. Using the notations for a general AKSZ model from subsection 2.1, we now briefly review the idea. First, the interaction part of the action, which we now call Θ , is an odd function on some even graded symplectic manifold (\mathcal{M}, ω) . The master equation implies $\{\Theta, \Theta\} = 0$, implying that the Hamiltonian vector field generated by Θ is a candidate for Q , $Q^B = (\Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB}$. For simplicity, we assume that the degrees are all carried by the coordinates Φ and ω is a constant of even degree. Moreover we assume that \mathcal{M} is a vector space (if this is not the case, we have to apply the exponential map). Following [6], we expand the equation

$$(\Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB} \partial_B \Theta = 0$$

into a power series around a fixed point $\Phi^A = \Phi_0^A + \phi^A$. By ordering the ϕ 's wisely, we have a neat expression without sign factors,

$$\begin{aligned} & 2(\phi^{C_1} \partial_{C_1} \Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB} \partial_B \Theta \\ & + (\phi^{C_2} \phi^{C_1} (\partial_{C_1} \partial_{C_2} \Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB} \partial_B \Theta + \phi^{C_2} (\partial_{C_2} \Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB} \phi^{C_1} \partial_{C_1} \partial_B \Theta) + \dots \\ & + \sum_{p=0}^n \frac{1}{p!(n-p)!} \phi^{C_p} \dots \phi^{C_1} (\partial_{C_1} \dots \partial_{C_p} \Theta \overleftarrow{\partial}_A)(\omega^{-1})^{AB} \phi^{C_n} \dots \phi^{C_{p+1}} (\partial_{C_{p+1}} \dots \partial_{C_n} \partial_B \Theta) \\ & + \dots, \end{aligned}$$

where all the derivatives are evaluated at the point Φ_0 . The coefficient of each power of ϕ has to vanish and we get a series of equations. By defining some short-hands

$$\Theta_m = \phi^{C_m} \dots \phi^{C_1} \frac{1}{m!} \partial_{C_1} \dots \partial_{C_m} \Theta(\Phi_0), \quad \Theta^m = \sum_{n=m}^{\infty} \Theta_n, \quad (4.1)$$

we can rewrite the equations as

$$\{\Theta_1, \Theta_2\} = 0, \quad \{\Theta_1, \Theta^3\} + \frac{1}{2} \{\Theta^2, \Theta^2\} = 0,$$

where $\{, \}$ are Poisson brackets in ϕ -variable. The first equation says $Q^A \partial_A Q^B = 0$, *i.e.* $Q^2 = 0$. The second is best rewritten as

$$Q^A \frac{\partial}{\partial \Phi_0^A} \Theta^2 + \frac{1}{2} \{\Theta^2, \Theta^2\} = 0. \quad (4.2)$$

This is essentially because, for example, at order 3

$$\{\Theta_1, \Theta_3\} = \Theta \overleftarrow{\partial}_B (\omega^{-1})^{BA} \frac{1}{2} \phi^C \phi^D \partial_D \partial_C \partial_A \Theta = Q^A \frac{1}{2} \phi^C \phi^D \partial_D \partial_C \partial_A \Theta = Q^A \frac{\partial}{\partial \Phi_0^A} \Theta_2. \quad (4.3)$$

This step is true for a flat target manifold, since the ordinary derivatives (graded) commute. For the curved case, the power series expansion is replaced with an exponential map that identifies a neighborhood of Φ_0 with the tangent space at Φ_0 . As a result the last rewriting must be checked by hand. For the BF-RW and CS-RW models on a hyperKähler manifold, however, (4.3) stand correct. We will discuss more about this issue of globalization in section 5, where we relax the hyperKähler condition.

Back from our digression, (4.2) is just the Maurer-Cartan equation and Θ^2 is a flat connection on $(\Phi_0, \phi) \in \mathcal{M} \times \mathcal{M}$ where in the ϕ -directions we have the bracket $\{\cdot, \cdot\}$ and the Φ_0 -directions we treat as a base.

The second ingredient for the Chern-Weil homomorphism is a cocycle. That will be provided by the path integral, more precisely by its perturbative expansion which is well defined.

We look at an AKSZ theory with only the kinetic term

$$\int_{\mathcal{L}} D\Phi \exp \left(\int d^6 z \Phi^A \omega_{AB} D\Phi^B \right) .$$

Here we assume that the Φ takes value in a flat symplectic manifold (\mathcal{M}, ω) (otherwise we use the exponential map for the expansion). For a collection of functions f_0, \dots, f_q on \mathcal{M} , consider the correlator,

$$c^q(f_0, f_1, \dots, f_q) = \int_{\mathcal{L}} D\Phi \int d^6 z_0 f_0(\Phi) \int d^6 z_1 f_1(\Phi) \dots \int d^6 z_q f_q(\Phi) \exp \left(\int d^6 z \Phi^A \Omega_{AB} D\Phi^B \right) \quad (4.4)$$

This correlator can be thought of as a cochain in the CE complex of the Lie algebra of Hamiltonian vector fields on \mathcal{M} . By using the property that the integral of some Δ -exact function over a Lagrangian submanifold is zero, and also the properties (2.3), (2.4) we have

$$\begin{aligned} 0 &= \int_{\mathcal{L}} D\Phi \Delta \left[\int d^6 z_0 f_0(\Phi) \int d^6 z_1 f_1(\Phi) \dots \int d^6 z_q f_q(\Phi) \exp \left(\int d^6 z \Phi^A \Omega_{AB} D\Phi^B \right) \right] \\ &= \sum_{i < j} (-1)^{f_i + s_{ij}} c^{q-1}(\{f_i, f_j\}, f_0, \dots, \hat{f}_i, \dots, \hat{f}_j, \dots, f_q) \\ &= \delta c^q(f_0, \dots, f_q) , \end{aligned} \quad (4.5)$$

where we abbreviate $(-1)^{\deg f_i}$ as $(-1)^{f_i}$ and define the Kozul sign

$$s_{ij} = (f_i + 1)(f_0 + \dots + f_{i-1} + i) + (f_j + 1)(f_0 + \dots + f_{j-1} + j) + (f_i + 1)(f_j + 1) .$$

The details of the derivation can be found in [6]. In conclusion, the cochain (4.4) provided by the path integral is a cocycle. It can be shown that if one changes the choice of \mathcal{L} , one only changes the cocycle by a coboundary.

This cocycle condition is derived when we perform the path integral over all the fields, but in practice, the integration of zero modes can be tricky. One can instead refrain from integrating over zero modes, leaving them as parameters. This way the cochains no longer take value in the real numbers but rather become cochains of the Lie algebra of Hamiltonian vector fields taking values in $C^\infty(\mathcal{M})$, and thereby giving a richer structure. In such cases,

the above cocycle condition is modified. Here we only write down the differential operator acting on a 1-cocycle for later use

$$\begin{aligned}\delta c^1(f, g, h) = & (-1)^f c^1(\{f, g\}, h) - (-1)^{f \deg c^1} \{f, c^1(g, h)\} \\ & + (-1)^{f+(h+1)(g+1)} c^1(\{f, h\}, g) - (-1)^{(g+1)(f+1)+g \deg c^1} \{g, c^1(f, h)\} \\ & + (-1)^{g+(f+1)(g+h)} c^1(\{g, h\}, f) - (-1)^{(h+1)(f+g)+h \deg c^1} \{h, c^1(f, g)\}\end{aligned}\quad (4.6)$$

In this expression, $\deg c^1$ is the degree of c^1 , namely $\deg c^1(f, g) = \deg f + \deg g + \deg c^1$. For the cochains of type (4.4), the degree is zero.

Now we have all the ingredients in the Chern-Weil homomorphism, a cocycle and a flat connection, and we can consider the perturbative expansion of the model defined by

$$S = \int d^6 z \left(\frac{1}{2} \Phi^A \omega_{AB} D \Phi^B + \Theta \right). \quad (4.7)$$

At each order of perturbation, we have a correlator

$$Z(\Phi_0) = \sum_{q=1}^{\infty} \hbar^q c^{2q-1}(\Theta^2, \dots, \Theta^2),$$

where Θ^2 is defined in (4.1). Each term c^{2q-1} in the expansion is Q -closed,

$$Q c^{2q-1}(\Theta^2, \dots, \Theta^2) = -q c^{2q-1}(\{\Theta^2, \Theta^2\}, \dots, \Theta^2) = \frac{1}{2(2q-1)} \delta c^{2q-1}(\Theta, \dots, \Theta) = 0$$

and furthermore, the change in \mathcal{L} only effects a change in Q -exact terms. We conclude that $c^{2q-1}(\Theta, \dots, \Theta) \in H_Q^{2q}(M)$. The class represented by c^{2q-1} is independent of the choice \mathcal{L} (note that all the extraneous data such as the metric of the target or source manifold comes in only through \mathcal{L}). The above discussion applies to a general AKSZ model.

Moreover the partition function can be written as a sum over graphs

$$Z(\Phi_0) = \sum_{\Gamma} b_{\Gamma} c_{\Gamma}(\Phi_0), \quad (4.8)$$

where b_{Γ} is given by integrals over correlators and $c_{\Gamma}(\Phi_0)$ is constructed by contracting $\partial_{C_1} \dots \partial_{C_m} \Theta(\Phi_0)$ by $(\omega^{-1})^{AB}$ according to the graph Γ . It is important that

$$\sum_{\Gamma} c_{\Gamma}(\Phi_0) \Gamma \quad (4.9)$$

is a graph cycle up to Q -exact terms. In fact the data (Θ, ω) on \mathcal{M} constitutes an L_{∞} algebra structure and for such a structure one can always construct graph cycles [16]. If we restrict ourselves only to trivalent graphs (this is what our perturbative theory naturally produces), then this statement says that $c_{\Gamma}(\Phi_0)$ satisfies IHX-relation up to Q -exact terms. Consequently, the partition function and therefore the characteristic classes (4.8) only depend on the graph cohomology class of $\sum_{\Gamma} b_{\Gamma} \Gamma^*$. More details about the interplay between b_{Γ} and c_{Γ} can be found in [6].

4.1 Characteristic Classes from the RW, BF-RW and CS-RW Models

For the RW model, recall that we have the MC equation (3.6), with $Q_{RW} = \bar{\partial}$ and the RW classes are elements of the Dolbeault cohomology. These class are related to the so called holomorphic foliation. For such cases, we have a principle bundle structure, the gauge group is formal holomorphic symplectomorphisms and its Lie algebra is formal holomorphic Hamiltonian vector fields. Here formal means that the Hamiltonian function that generates these vector fields are formal power series in ξ^i . It also turns out that our function $\Theta_{\bar{i}}(\xi)$ is the flat connection arising from the foliation problem as prescribed by Fuks in [15, 17], the algorithm is clearly laid out in the latter reference. The same procedure is come upon independently by a number of authors [18, 14], in [14] it is also explained the need to restrict the cocycle to be basic, that is, the cocycle is invariant under $sp(2n)$ transformations, and if any of the functions f_i in (4.4) is quadratic, the cochain vanishes. Both properties are clearly satisfied by our path integral construction and are crucial for the later analysis of invariance properties of our characteristic classes. This interpretation of the RW invariants was outlined in [2], further clarified in [3, 6].

4.2 BF-RW Model

We can slightly generalize the previous construction to incorporate a gauge field. For the RW model on a hyperKähler manifold coupled to BF theory, defined in (2.14), we would like to apply the same exponential map (3.4). The resulting action is

$$S = \int d^6 z \left(B_A D A^A + p_{\bar{i}} D X^{\bar{i}} + q_{\bar{i}} D v^{\bar{i}} + \frac{1}{2} \Omega_{ij} \xi^i D \xi^j + \frac{1}{2} f_{AB}^C A^A A^B B_C - p_{\bar{i}} v^{\bar{i}} + v^{\bar{i}} \Theta_{\bar{i}} + M_A^0 A^A \right),$$

where $\Theta_{\bar{i}}$ is defined in (3.5) and

$$M_A^n = \sum_{k=n}^{\infty} M_{Ak}, \quad M_{Ak} = \frac{1}{k!} \xi^{i_1} \cdots \xi^{i_k} \nabla_{i_k} \cdots \nabla_{i_2} \partial_{i_1} \mu_A(x_0), \quad (4.10)$$

where we use the hyperKähler properties. Some of its properties are

$$\bar{\partial} M_A^2 = \bar{\partial} M_A^1 = \bar{\partial} M_A^0 = -\{v^{\bar{i}} \Theta_{\bar{i}}, M_A^1\}; \quad \{A^A M_A^1, A^B M_B^1\} = A^A A^B f_{AB}^C M_C^0. \quad (4.11)$$

We need to derive a Maurer-Cartan equation in this case. One can of course follow the procedure as for (4.2), but we would like to also treat the gauge fields as parameters before choosing a gauge fixing. We need to modify somewhat the procedure that leads to (4.2). To this end, assume that the symplectic form of the target space splits into two independent parts $\omega = \omega_p + \Omega$, where ω_p is for the 'parameter sector' and Ω is for the fields that are active in perturbation (*i.e.* only ξ). Again expanding the equation

$$\{\Theta, \Theta\}_{\omega_p} + \{\Theta, \Theta\}_{\Omega} = 0,$$

into a power series of the *active fields only*, we obtain the equation series

$$\begin{aligned} \{\Theta^0, \Theta^0\}_{\omega_p}^1 + 2\{\Theta_1, \Theta^2\}_\Omega + \{\Theta^2, \Theta^2\}_\Omega &= 0, \\ \{\Theta^0, \Theta^0\}_{\omega_p}^2 + 2\{\Theta_1, \Theta^3\}_\Omega + \{\Theta^2, \Theta^2\}_\Omega &= 0, \\ \{\Theta^0, \Theta^0\}_{\omega_p}^3 + 2\{\Theta_1, \Theta^4\}_\Omega + 2\{\Theta_2, \Theta^3\}_\Omega + \{\Theta^3, \Theta^3\}_\Omega &= 0. \end{aligned} \quad (4.12)$$

Pay attention that the numbers in the super or subscript only denote the powers of the active fields, not the parameters. This is the Maurer-Cartan equation modified due to our needs to treat certain fields as parameters. We have chosen to go up to fourth order, because we know that eventually the vertices appearing in the perturbation will be of cubic order.

For the BF-RW model, we treat $p_{\bar{i}}, X^{\bar{i}}, q_{\bar{i}}, v^{\bar{i}}, A, B$ as parameters, while ξ^i alone is active. Further notice ⁶

$$\begin{aligned} \{\Theta^0, \Theta^0\}_{\omega_p}^3 &= 2\bar{\partial}(v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3) - A^A A^B f_{AB}^C M_C^3 = 2(\bar{\partial} - \frac{1}{2}A^A A^B f_{AB}^C \frac{\partial}{\partial A^C})(v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3), \\ \{\Theta_1, \Theta^4\}_\Omega &= \{A^A(\partial_i \mu_A)\xi^i, v^{\bar{i}}\Theta_{\bar{i}}^4 + A^A M_A^4\} = A^A k_A^i \nabla_{x_0^i}(v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3), \end{aligned}$$

and we obtain our modified Maurer-Cartan equation

$$\begin{aligned} Q(v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3) + \frac{1}{2}\{v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3, v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3\} + \{A^A M_{A2}, v^{\bar{i}}\Theta_{\bar{i}} + A^A M_A^3\} &= 0, \\ Q &= (\bar{\partial} + A^A k_A^i \nabla_{x_0^i} - \frac{1}{2}A^A A^B f_{AB}^C \frac{\partial}{\partial A^C}), \end{aligned} \quad (4.13)$$

where the brackets $\{, \}$ are taken in ξ -direction and in Q the covariant derivative $\nabla_{x_0^i}$ can be replaced by the partial derivative when it acts on $(0, p)$ -forms. So the MC equation is satisfied up to a $sp(2n)$ rotation in the ξ^i space (the last term of the first line). Yet from our earlier discussion, the cochain defined by the path integral is invariant with respect to $sp(2n)$ rotations. Also, Q does not square to zero but rather a curvature term. This is again not a problem, since for our purpose, Q will always act on $(0, p)$ forms while the connection is of pure index. Finally, if one does not trust the above derivation, by using the property of M in (4.11) one can check that the MC equation is indeed fulfilled.

Proceeding to the lowest order of the perturbative expansion, the Feynman rules gives for the following graph

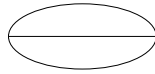


Figure 2: Γ , perturbation expansion to the lowest order

$$c^1(\Theta, \Theta) = \frac{1}{6}(\Theta \overleftarrow{\partial}_{A_1} \overleftarrow{\partial}_{A_2} \overleftarrow{\partial}_{A_3})(\Omega^{-1})^{A_1 B_1}(\Omega^{-1})^{A_2 B_2}(\Omega^{-1})^{A_3 B_3}(\partial_{B_1} \partial_{B_2} \partial_{B_3} \Theta). \quad (4.14)$$

⁶Due to unfortunate notation, there is a likely confusion between Θ , which is a generic interaction term for the action (4.7), and $\Theta_{\bar{i}}$ defined in (3.5).

If we specialize to our BF-RW theory, the interaction term $\Theta = v^{\bar{i}}\Theta_{\bar{i}} + A^A\mathbf{M}_A^3$ plays the role of the 3-valent vertices and we obtain

$$\begin{aligned} c_\Gamma(x_0, v, A) &= \frac{1}{6}\partial_{\bar{i}}\partial_j\partial_k(v^{\bar{i}}\Theta_{\bar{i}} + A^A\mathbf{M}_A^3)(\Omega^{-1})^{jo}(\Omega^{-1})^{kp}(\Omega^{-1})^{lq}\partial_o\partial_p\partial_q(v^{\bar{i}}\Theta_{\bar{i}} + A^A\mathbf{M}_A^3) \\ &= \frac{1}{6}(R_{\bar{i}j}{}^n{}_l v^{\bar{i}}\Omega_{nk} + A^A\nabla_j\nabla_k\partial_l\mu_A)(\Omega^{-1})^{jo}(\Omega^{-1})^{kp}(\Omega^{-1})^{lq}(R_{\bar{i}o}{}^r{}_q v^{\bar{i}}\Omega_{rp} + A^A\nabla_o\nabla_p\partial_q\mu_A) . \end{aligned} \quad (4.15)$$

From the general arguments given in the beginning of section 4, we have $Qc_\Gamma = 0$. Indeed one can check explicitly that $Qc_\Gamma = 0$ using the above concrete expressions for Q and c_Γ . This is a straightforward but lengthy calculation. c_Γ is merely a representative of the Q -cohomology class when one uses the hyperKähler metric. When using a general metric, the expression for c_Γ will be much more complicated, but still lies in the same Q -class.

4.3 CS-RW Model

In this case, the gauge fields \mathbf{A} does take part in the perturbation theory actively, since now the propagators in the gauge sector is $\langle \mathbf{A}(x, \theta), \mathbf{A}(y, \xi) \rangle$ in contrast to $\langle \mathbf{A}(x, \theta), \mathbf{B}(y, \xi) \rangle$ in BF theory. The application of the exponential map is exactly the same as before, and we follow the same procedure as in the previous section to derive the Maurer-Cartan equation. The gauge field is now split as $\mathbf{A} = \mathbf{A}^h + \mathbf{A}^c$, the superscripts h, c suggest harmonic and coexact. Let us abuse the notation a little and use A^h as base coordinate and A^c as fiber coordinate in the Taylor expansion (see section 4 for the expansion into a power series around a fixed point $\Phi^A = \Phi_0^A + \phi^A$). Note that the expansion is slightly different

$$\begin{aligned} \Theta^2 &= (A^A)^h\mathbf{M}_A^2 + (A^A)^c\mathbf{M}_A^1 + \frac{1}{2}f_{ABC}(A^A)^h(A^B)^c(A^C)^c, \\ \Theta^3 &= v^{\bar{i}}\Theta_{\bar{i}} + (A^A)^h\mathbf{M}_A^3 + (A^A)^c\mathbf{M}_A^2 + \frac{1}{6}f_{ABC}(A^A)^c(A^B)^c(A^C)^c . \end{aligned}$$

The Maurer-Cartan equation reads

$$\begin{aligned} \bar{\partial}\Theta^3 + \left(-\frac{1}{2}f_{ABC}(A^A)^h(A^B)^h\frac{\partial}{\partial(A^C)^h} - \mu_A\eta^{BA}\frac{\partial}{\partial(A^A)^h} + (A^A)^hk_A^i\nabla_i\right)\Theta^3 \\ + \{\Theta_2, \Theta^3\} + \frac{1}{2}\{\Theta^3, \Theta^3\} = 0 , \end{aligned} \quad (4.16)$$

where the bracket $\{ , \}$ is understood in ξ - and A -directions. There are two features to this equation. Firstly it is no longer homogeneous in powers of A and v ; secondly the next to last term now represents an $osp(2n|m)$ rotation where $2n$ is the complex dimension of the manifold and m is the rank of the gauge group.

For the graph fig.2, the Feynman rules give (we have dropped the superscript h on A , for it is clear that the A 's that are uncontracted must be the harmonic modes)

$$\begin{aligned} c_\Gamma(x_0, v, A) &= \frac{1}{6}(R_{\bar{i}j}{}^n{}_l v^{\bar{i}}\Omega_{nk} + A^A\nabla_j\nabla_k\partial_l\mu_A)(\Omega^{-1})^{jo}(\Omega^{-1})^{kp}(\Omega^{-1})^{lq}(R_{\bar{i}o}{}^r{}_q v^{\bar{i}}\Omega_{rp} + A^A\nabla_o\nabla_p\partial_q\mu_A) \\ &\quad + \frac{1}{2}\nabla_j\partial_k\mu_A\nabla_m\partial_n\mu_B\Omega^{jm}\Omega^{kn}\eta^{AB} . \end{aligned}$$

Again, by general arguments this expression should be Q -closed, where Q is given by

$$Q = \bar{\partial} - \frac{1}{2} f_{ABC} A^A A^B \frac{\partial}{\partial A^C} + \mu_A \eta^{BA} \frac{\partial}{\partial A^A} + A^A k_A^i \nabla_i , \quad (4.17)$$

where the covariant derivative can be replaced by ordinary ones when it acts on $(0, p)$ -forms. Again, it is a straightforward but lengthy calculation to check this explicitly.

4.4 Some Explicit Checks

Now let us return to a discussion of general 3D AKSZ theories. We can argue that any trivalent graph gives rise to a cocycle of Hamiltonian vector fields. First, let us again perform some quick checks on the statement $Q_{\text{cr}} = 0$ from another perspective, and see how it can be generalized for any trivalent graph. As mentioned above, the Maurer-Cartan equations (4.2), (4.16) are rather easy to check by using the properties (3.6), (4.11). What remains to be checked stands thus

$$0 \stackrel{?}{=} Qc^1(\Theta^3, \Theta^3) = -2c^1(\{\Theta_2, \Theta^3\}, \Theta^3) - 2c^1(\{\Theta^3, \Theta^3\}, \Theta^3) .$$

The first term on the right hand side is zero due to the $sp(2n)$ or $osp(2n|m)$ invariance of the cochain c^1 . For the second term, by using the definition of the differential operator (4.6), it can be written as

$$\delta c^1(\Theta^3, \Theta^3, \Theta^3) = -3c^1(\{\Theta^3, \Theta^3\}, \Theta^3) - 3\{\Theta^3, c^1(\Theta, \Theta)\} = -3c^1(\{\Theta^3, \Theta^3\}, \Theta^3) .$$

The last term drops because the operation $\{\Theta^3, \cdot\}$ requires Θ^3 to have non-vanishing first derivative in ξ at $\xi = 0$, but Θ^3 starts off at order ξ^3 .

So we only need to check whether the cochain c^1 is a cocycle. In the following, we explicitly check the case where all fields involved are purely bosonic, yet enough detail is given so that the reader can check the general case. The cochain c^1 , when evaluated on two functions, gives

$$c^1(f, g) = (\partial_A \partial_B \partial_C f)(\omega^{-1})^{AD} (\omega^{-1})^{BE} (\omega^{-1})^{CF} (\partial_D \partial_E \partial_F g) ,$$

and the cocycle condition reads

$$c^1(\{f, g\}, h) - \{f, c^1(g, h)\} + \text{cyc perm} = 0 ,$$

where for the sake of clarity we assume that f, g, h and the coordinates Φ^A are purely bosonic. For the general graded case the argument can be repeated.

Even for this simple formula, it is not recommended to check it by brute force. We can use a graphic method to simplify things. We denote $(\partial_A f)(\omega^{-1})^{AB} (\partial_B g)$ as $f \rightarrow g$; $(\partial_A \partial_B f)(\omega^{-1})^{AC} (\omega^{-1})^{BD} (\partial_C \partial_D g)$ as $f \Rightarrow g$, etc. Reversing an arrow means a minus sign. The cocycle condition amounts to

$$\left((f \rightarrow g) \Rightarrow h \right) - \left(f \rightarrow (g \Rightarrow h) \right) + \text{cyc perm} = 0 .$$

Then easily we get (the over-arrow is meant to go from f to h)

$$\begin{aligned}
& \left((f \rightarrow g) \Rightarrow h \right) - \left(f \rightarrow (g \Rightarrow h) \right) \\
= & (g \leftarrow f \Rightarrow h) + (f \rightarrow g \Rightarrow h) - (f \rightarrow g \Rightarrow h) - (\overrightarrow{f \rightarrow g \Rightarrow h}) \\
& + 3(\overrightarrow{f \rightarrow g \Rightarrow h}) + 3(\overrightarrow{f \Rightarrow h \leftarrow g}) ,
\end{aligned}$$

each of the terms on the right hand side will find their companion with a minus sign from the cyclic permutations.

In fact, using this graph method, it is not difficult to show that if a cochain is represented solely by 3-valent graphs, then it is closed. This result probably is already known to Kontsevich [16]. We stress that this does not trivialize our result, for what is more interesting than the closedness of these cochains is how they change under a change of gauge condition and metric etc. These properties follows non-trivially from the BV manipulations.

5 Mathematical Comments

In this section we collect some mathematical comments on the RW, BF-RW and CS-RW models. In particular we would like to explain briefly the general case of a holomorphic symplectic manifold. We present the construction of the corresponding characteristic classes and argue their independence from the connection used in the construction.

In the earlier subsection 2.1, we took the AKSZ action

$$S_{RW} = \int d^6 z \left(\mathbf{p}_{\bar{i}} D \mathbf{X}^{\bar{i}} + \mathbf{q}_{\bar{i}} D \mathbf{v}^{\bar{i}} + \frac{1}{2} \Omega_{ij} \mathbf{X}^i D \mathbf{X}^j - \mathbf{p}_{\bar{i}} \mathbf{v}^{\bar{i}} \right) ,$$

whose definition only depended on the complex structure and holomorphic symplectic form, and we specialized to the case of a hyperKähler manifold and obtained the RW classes.

Since the metric of the hyperKähler manifold does not enter the defining data of the action, but rather comes in later through the exponential map (*i.e.* through the choice of gauge fixing), by a general physical argument, we know that the RW classes do not depend on the hyperKähler metric. In contrast, in the work by Kapranov [3] and reviewed by Sawon [4], the authors chose to integrate the RW class on the hyperKähler manifold and obtain the RW invariants. After performing the integral, a somewhat stronger invariance property can be claimed. The key observation is that the 3-valent vertex in the perturbation theory is identified as the Atiyah class, which is the obstruction to the existence of a holomorphic connection. The Dolbeault representative of this class is just the (1,1) component of the curvature tensor. But for a holomorphic vector bundle with Hermitian metric one can always construct a connection such that the curvature is automatically (1,1). Thus the Atiyah class can be defined without reference to the complex structure. Furthermore, this class is by construction independent of the Kähler form ω , and hence independent of $\text{Re}\Omega$ and $\text{Im}\Omega$ by the total democracy between ω , $\text{Re}\Omega$, $\text{Im}\Omega$. This last statement leads to the claim that the RW invariants are constant on the connected component of the moduli

space of hyperKähler metrics. But from our approach, without integrating the RW class, the last claim can not be made.

On the other hand, we will now relax the hyperKähler condition. From the earlier discussion of perturbation theory of the RW model, we saw a few salient features. Firstly, we needed to expand around a fixed point x_0 using the exponential map. It was crucial that there the Levi-Civita connection preserves Ω , the symplectic form $\Omega_{ij}(X)\delta X^i\delta X^j$ is pulled back neatly into $\Omega_{ij}(x_0)\delta\xi^i\delta\xi^j + 2\delta\Theta_{\bar{i}}\delta X^{\bar{i}}$, *i.e.* after a shift of $p_{\bar{i}}$ that absorbs the second term, the ξ fields live in \mathbb{C}^{2n} with a constant symplectic form. Thus our simple gauge fixing condition $\xi^e = 0$ can be applied. Secondly, to obtain the relevant Maurer-Cartan equation, we pulled back the $\bar{\partial}$ and showed that it is given by $\bar{\partial} + v^{\bar{i}}\{\Theta_{\bar{i}}, \cdot\}$ in the pull back frame. Once we relax the hyperKähler condition, we still would like to preserve these features, for otherwise, the structure of perturbation theory changes drastically and we can no longer make reliable claims about metric independence.

For an integrable complex structure, it is known that one can construct a torsionless connection, such that $\nabla J = 0$ (e.g., see [19]). Such a connection, written in complex coordinates, has the only non-zero components $\Gamma_{\mu j}^i$ or $\Gamma_{\mu \bar{j}}^{\bar{i}}$. Then by the torsionless condition $\Gamma_{il}^j = \Gamma_{li}^j = 0$, *i.e.* Γ is of purely holomorphic or anti-holomorphic indices. Based on this connection, we can further construct another one such that Ω is preserved. Let

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{3}\Omega^{kl}(\nabla_i\Omega_{lj} + \nabla_j\Omega_{li}) .$$

The new connection has the same transformation property as the original one, since the additional piece is covariant. It is torsionless and preserves Ω

$$\begin{aligned} \tilde{\nabla}_i\Omega_{jk} &= \nabla_i\Omega_{jk} - \frac{1}{3}\Omega^{lm}(\nabla_i\Omega_{m[j} + \nabla_{[j}\Omega_{|mi})\Omega_{l|k]} \\ &= \nabla_i\Omega_{jk} + \frac{1}{3}(2\nabla_i\Omega_{kj} + \nabla_{[j}\Omega_{k]i}) = \frac{1}{3}\nabla_i\Omega_{jk} + \text{cyc} \\ &= \frac{1}{3}(d\Omega)_{ijk} = 0 . \end{aligned}$$

Note that in the last step, the torsionless condition for Γ is used to convert covariant derivatives to ordinary derivatives.

We shall drop the \sim henceforth and perform the exponential map using the new connection⁷. To obtain the Maurer-Cartan equation, we need to pull back $\bar{\partial}$. The procedure is in [6], one tries to find a Θ such that

$$\exp_{\xi}(\bar{\partial} + \{\Theta, \cdot\})\exp_{\xi}^{-1} = \bar{\partial} .$$

The left hand side can be expanded by using the formula

$$\begin{aligned} \exp(-\xi \cdot \nabla)\mathcal{O}\exp \xi \cdot \nabla &= \mathcal{O} - [\xi \cdot \nabla, \mathcal{O}] + \frac{1}{2}[\xi \cdot \nabla, [\xi \cdot \nabla, \mathcal{O}]] + \dots , \\ \xi \cdot \nabla &= \xi^i\partial_{x^i} - \xi^i\xi^j\Gamma_{ij}^k\partial_{\xi^k} , \end{aligned}$$

⁷The use of such a connection for the RW model was previously suggested also in [1] and [20].

where ∂_{x^i} denotes the derivative with respect to the base point of the expansion.

Suppose that the curvature tensor of Γ is (1,1), then this infinite sum can be worked out explicitly, and one obtains (3.5). That Θ satisfies the MC equation follows from the construction. In the general case, we cannot argue that the curvature of Γ is of type (1,1) (this would have been true had we used the unique Hermitian connection associated with a Hermitian metric). As a result, starting from order ξ^4 , there will be a new term $\xi^{l_1}\xi^{l_2}\xi^{l_3}\xi^{l_4}(R_{\bar{i}l_1}^k R_{kl_3}^i) \nabla_i$ in the expansion of $\exp(-\xi \cdot \nabla) \bar{\partial} \exp \xi \cdot \nabla$. Due to this complication, we cannot work out the exponential map to all orders as in (3.5). However, as far as the perturbation theory is concerned, we only require the knowledge of $\Theta_{\bar{i}}$ up to order ξ^3 , and the 3-valent vertex remains the same

$$V_{ijk} = v^{\bar{i}} R_{\bar{i}i}^l \Omega_{lj} , \quad (5.1)$$

which is automatically symmetric in i, j, k if ∇ preserves Ω . The MC equation up to this order is satisfied trivially $\bar{\partial} V = 0$; the rest of the story of the perturbation theory follows through, and in conclusion, we are able to construct RW classes for a holomorphic symplectic manifold. The choice of the connection, since such connection is not unique, will not affect the RW class using a path integral argument. In fact, this follows rather trivially from the fact that the vertex function is $\bar{\partial}$ -closed: $\bar{\partial}_{[\bar{i}} R_{\bar{j}]j}^k = \nabla_{[\bar{i}} R_{\bar{j}]j}^k = \partial_j R_{\bar{i}\bar{j}}^k = 0$, and that its variation is $\bar{\partial}$ -exact: $\delta R_{\bar{i}j}^k = \bar{\partial}_{\bar{i}} \delta \Gamma_{j\bar{i}}^k$. Therefore we see that under a change of connection the RW class does not change.

Similar logic can be applied to the BF-RW, though in this case things work out quite nontrivially. As said above, the exponential map cannot be worked out to all orders. To order ξ^2 ,

$$\begin{aligned} \exp^* \frac{1}{2} \Omega_{ij} \delta X^i \delta X^j &= \frac{1}{2} \Omega_{ij} \delta \xi^i \wedge \delta \xi^j - \frac{1}{2} \delta X^{\bar{i}} \wedge \delta \xi^j (R_{\bar{i}m}^i \xi^m \xi^n) \\ &\quad - \frac{1}{6} \delta \xi^i \wedge \delta \xi^j (R_{im}^q \Omega_{qj} \xi^m \xi^n) + \mathcal{O}(\xi^3) . \end{aligned}$$

We emphasize that the Ω appearing here is a constant $\Omega(x_0)$. We have to bring the symplectic form to canonical form, namely $\omega = \delta p_{\bar{i}} \delta X^{\bar{i}} + 1/2 \Omega_{ij} \delta \xi^i \wedge \delta \xi^j + \delta q_{\bar{i}} \delta v^{\bar{i}}$. We already know how to reshuffle the second term: up to order ξ^2 , it can be absorbed by a shift

$$\tilde{p}_{\bar{i}} = p_{\bar{i}} + \frac{1}{6} R_{\bar{i}i}^l \Omega_{lj} \xi^i \xi^j \xi^k + \mathcal{O}(\xi^3) ,$$

this is the same shift we have seen in the RW model. Now for the third term, it can be written as

$$- \frac{1}{6} \delta \xi^i \wedge \delta \xi^j (R_{im}^q \Omega_{qj} \xi^m \xi^n) = \frac{1}{24} \delta (R_{ij}^n \Omega_{nk} \xi^j \xi^k \xi^l) \wedge \delta \xi^i .$$

We therefore make a shift of ξ

$$\tilde{\xi}^i = \xi^i - \frac{1}{24} \Omega^{ip} R_{pj}^n \Omega_{nk} \xi^j \xi^k \xi^l + \mathcal{O}(\xi^4) .$$

Let us investigate the consequence of the second shift. After the exponential map and the first shift, the interaction term becomes

$$\begin{aligned} -p_{\bar{i}}v^{\bar{i}} + A^A\mu_A + \frac{1}{2}f_{AB}^CA^AB^BB_C = & -\tilde{p}_{\bar{i}}v^{\bar{i}} + v^{\bar{i}}\frac{1}{6}R_{\bar{i}\bar{i}}^{\bar{l}}\Omega_{\bar{l}j}\xi^i\xi^j\xi^k + \frac{1}{2}f_{AB}^CA^AA^BB_C \\ & + A^A\left(\mu_A(x_0) + (\partial_i\mu_A)\xi^i + \frac{1}{2}(\nabla_i\partial_j\mu_A)\xi^i\xi^j + \frac{1}{6}(\nabla_i\nabla_j\partial_k\mu_A)\xi^i\xi^j\xi^k\right) + \mathcal{O}(\xi^4) . \end{aligned}$$

We drop the tilde on $p_{\bar{i}}$ as usual. The second shift only affects the second term in the round brace, since the shift is of order ξ^3 and we are working up to order ξ^3 . So we get a new term

$$(\partial_i\mu_A)\xi^i \rightarrow (\partial_i\mu_A)\tilde{\xi}^i + \frac{1}{24}k_A^iR_{ij}^n\Omega_{nk}\tilde{\xi}^j\tilde{\xi}^k\tilde{\xi}^l \quad (5.2)$$

for our 3-valent vertex. This unexpected term is in fact crucial for the connection independence of the theory. To see this, suppose we choose a different connection (still preserving J, Ω) $\Gamma \rightarrow \Gamma + \delta\Gamma$, then the variation of the interaction term of the action with respect to the connection is⁸

$$\begin{aligned} \delta_\Gamma S_{int} &= \{\Psi, S_{int}\} , \\ \Psi &= -\frac{1}{6}(\delta\Gamma_{mn}^i\Omega_{ip})\tilde{\xi}^m\tilde{\xi}^n\tilde{\xi}^p - \frac{1}{24}(\nabla_m\delta\Gamma_{nq}^i)\Omega_{ip}\tilde{\xi}^m\tilde{\xi}^n\tilde{\xi}^p\tilde{\xi}^q . \end{aligned} \quad (5.3)$$

The fact that $\delta_\Gamma S_{int}$ can be written as a canonical transformation should not be surprising. Because using either Γ or $\Gamma + \delta\Gamma$ for the expansion, the original symplectic form $\delta X^i\Omega_{ij}\delta X^j$ is pulled back to the same $\delta\tilde{\xi}^i\Omega_{ij}(x_0)\delta\tilde{\xi}^j$, implying the existence of a canonical transformation. Note that (5.3) fails without the extra term in (5.2).

Let us proceed to the analysis of the MC equation as we did in (4.2). If (4.2) stood, then our partition function would be Q -closed, and (5.3) would guarantee the connection independence. However, in this case (4.3) fails due to the fact $R_{ij}^k{}_l \neq 0$. But having (4.3) at hand is very desirable because it allows us to interpret our partition function as Q -cohomology classes. Here we present a simple solution to this problem. We see that the failure of (4.3) is a mismatch between $Q\Theta^3$ and $\{\Theta_1, \Theta^4\}$ (using the notation of that section), we can fix this mismatch by hand through modifying Θ order by order. It turns out that in our case the required modification at order $\tilde{\xi}^3$ is an extra $1/8A^Ak_A^iR_{ij}^n\Omega_{nk}\tilde{\xi}^j\tilde{\xi}^k\tilde{\xi}^l$.

To conclude, for the BF-RW model, the modified 3-valent vertex is

$$V = v^{\bar{i}}\Theta_{\bar{i}3} + A^AM_{A3} + A^AU_A ; \quad U_A = \frac{1}{6}k_A^iR_{ij}^n\Omega_{nk}\tilde{\xi}^j\tilde{\xi}^k\tilde{\xi}^l ,$$

where $\Theta_{\bar{i}3}, M_{A3}$ are defined in (3.5) and (4.10), but now with $\tilde{\xi}$ in place of ξ . We find that, after quite a nasty bit of calculation, up to order $\tilde{\xi}^3$, the vertex V does satisfy the MC equation,

$$QV + \{A^AM_{A2}, V\} = 0,$$

⁸Since Ψ is at least cubic, this is why the 3-valent vertex for the pure RW model (5.1) is not affected.

where Q is as in (4.13) and the bracket $\{ , \}$ is in $\tilde{\xi}$ -direction with respect to constant $\Omega(x_0)$.

To verify the connection independence, we vary V with respect to the connection Γ ,

$$\delta_\Gamma V = \left(\bar{\partial} + \{A^A \mathbf{M}_{A2}, \cdot\} + A^A k_A^l \nabla_l \right) W = QW + \{A^A \mathbf{M}_{A2}, W\} , \quad W = \frac{1}{6} \delta \Gamma_{ik}^n \Omega_{nj} \xi^i \xi^j \xi^k .$$

The variation of the BF-RW class is

$$\delta_\Gamma c^q(\underbrace{V, \dots V}_{q+1}) = (q+1) c^q(QW + \{A^A \mathbf{M}_{A2}, W\}, \underbrace{V, \dots V}_q) ,$$

where c^q is the cocycle given by the path integral.

$$\begin{aligned} \delta_\Gamma c^q(V, \dots V) &\sim -q(q+1) c^q(W, QV, V, \dots V) + (q+1) c^q(\{A^A \mathbf{M}_{A2}, W\}, V, \dots V) \\ &= q(q+1) c^q(W, \{A^A \mathbf{M}_{A2}, V\}, V, \dots V) + (q+1) c^q(\{A^A \mathbf{M}_{A2}, W\}, V, \dots V) , \end{aligned}$$

where \sim means the dropping of a Q -exact term. In the end, we see that the remaining terms combine to become an overall $sp(2n)$ rotation, and hence vanish.

In summary, we have given the prescription of constructing the equivariant RW-class associated with a holomorphic symplectic manifold, and demonstrated its invariance under the choice of connection. We believe that the way we fixed the problem above should originate from some deeper physical (mathematical) principles, which are probably related to the globalization issues discussed in [21] and the application of the Fedosov connection for handling perturbation theory on curved manifolds [22]. With all these said, our result as it stands is nonetheless rigorous and impregnable.

6 Summary and Outlook

We have in this paper provided a systematic construction of the RW, BF-RW and CS-RW models. We have shown that by applying the exponential map we can use a simple superfield computation to get exactly the same results as in the component approach. As an application of these theories, we showed that the evaluation of the partition function at each order of \hbar is none other than applying the Chern-Weil homomorphism. This way, we constructed certain characteristic classes as a generalization of the original Rozansky-Witten classes.

There are a few interesting points in the final form of the characteristic classes. Firstly, looking at (4.15), we observe the moment map does not appear un-differentiated. Thus these classes can be expressed using the vector field k_A^i instead. Thus the vertex looks as follows

$$V_{jkl} = dz^{\bar{i}} R_{\bar{i}j}^n \Omega_{nk} + A^A (\nabla_j \nabla_k k_A^n) \Omega_{nl} + A^A k_A^i R_{ij}^n \Omega_{nk} , \quad (6.1)$$

where symmetrization in j, k, l is assumed. This suggests that our result is also valid for a group action that preserves Ω , not necessarily Hamiltonian. Thus our result is applicable to much wider situations. Secondly, as we mentioned in the text, when we

use S^3 as our source manifold, there exists the brute force gauge fixing that reduces all Feynman diagrams to only 3-valent. The cochains arising from such Feynman diagrams are automatically closed. On the other hand we know that there exist CE cocycles that are not represented by 3-valent graphs, so using AKSZ TFT on S^3 , we will not reach any of the higher (and perhaps more interesting) classes. Yet, in our construction of these characteristic classes, the 3-manifold plays an entirely passive role. In general one is allowed to replace the de Rham complex on the 3-manifold with any differential graded Frobenius algebra (see [23, 24] for the construction) and the BV argument for the cocycle condition goes through just the same. It is therefore interesting to investigate this possibility and produce some higher classes, whose closedness is more genuinely dependent upon the path integral construction.

Last but not least, we provided an explicit formulae for the equivariant BF-RW classes on a holomorphic symplectic manifold. A physics based argument leads us to a vertex with a somewhat unexpected extra term, the existence of which is crucial for the invariance of the characteristic classes.

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